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ORIGINAL ARTICLE

Interval-valued Fuzzy Matrices with Interval-valued Fuzzy Rows and Columns

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Abstract Fuzzy matrix (FM) is a very important topic of fuzzy algebra. In FM, the elements belong to the unit interval [0, 1]. When the elements of FM are the subintervals of the unit interval [0, 1], then the FM is known as interval-valued fuzzy matrix (IVFM). In IVFM, the membership values of rows and columns are crisp, i.e., rows and columns are certain. But, in many real life situations they are also uncertain. So to model these types of uncertain problems, a new type of interval-valued fuzzy matrices (IVFMs) are called interval-valued fuzzy matrices with interval-valued fuzzy rows and columns (IVFMFRCs). For these matrices, null, identity, equality, etc. are defined along with some binary operators. Complement and density of IVFMFRC are defined and several important properties are investigated. An application of IVFM-FRC in image representation is also given.

Keywords Fuzzy matrix · Interval-valued fuzzy matrix · Fuzzy rows and columns · Complement of fuzzy matrices · Density of fuzzy matrices

1. Introduction

Like classical (crisp) matrices, fuzzy matrices (FMs) are now a very rich topic in modeling uncertain situations occurred in science, automata theory, logic of binary

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relations, medical diagnosis, etc. In FMs, only the elements are uncertain, while rows and columns are certain. But, in many real life situations we observed that rows and columns may also be uncertain. Pal [29] has defined fuzzy matrices with fuzzy rows and columns (FMFRCs).

The elements of FMFRCs are non-negative proper fraction. But, when the elements are the subintervals of the unit interval $[0, 1]$, then the FM is known as IVFM. In IVFM, the rows and columns are considered as scrips, but we have seen that they may also be uncertain, i.e., rows and columns have some membership values.

For example, in interval-valued fuzzy graph, the vertices and edges are uncertain and their membership values are the subintervals of $[0, 1]$. This graph can be represented as an IVFM. For each vertex there is a row and column. The ijth entry represents the membership value of the edge joining i and j. In this matrix, the rows and columns are uncertain.

Thomson [37] defined FMs for the first time in 1977. In this paper, he discussed about the convergence of the powers of a fuzzy matrix. The theories of fuzzy matrices were developed by Kim and Roush [22] as an extension of Boolean matrices. With max-min operation the fuzzy algebra and its matrix theory are considered by many authors [5, 14, 20, 26, 27, 34]. Hashimoto [15] studied the canonical form of a transitive fuzzy matrix. Xin [38] studied controllable fuzzy matrices. Hemashina et al. [18] investigated iterates of fuzzy circulant matrices. The transitivity of matrices over path algebra (i.e., additively idempotent semiring) is discussed by Hashimoto [15-17]. Generalized fuzzy matrices, matrices over an incline and some results about the transitive closure, determinant, adjoint matrices, convergence of powers and conditions for nilpotency are considered by Duan [13] and Lur et al. [23]. Dehghan et al. [12] give two ideas for finding the inverse of a fuzzy matrix viz. scenario-based and arithmetic-based.

There are some limitations in dealing with uncertainties by fuzzy sets. To overcome these difficulties, Atanassov [4] introduced the theory of intuitionistic fuzzy set in 1993 as a generalization of fuzzy sets. Based on this concept Pal et al. have defined intuitionistic fuzzy determinant in 2001 [26] and intuitionistic fuzzy matrices (IFMs) in 2002 [27]. Bhowmik and Pal [5] produced some results on IFMs, intuitionistic circulant fuzzy matrix and generalized intuitionistic fuzzy matrix [5-11]. Shyamal and Pal [33, 35] defined the distances between IFMs and hence defined a metric on IFMs. They also mentioned few applications of IFMs. In [25], the similarity relations, invertibility conditions and eigenvalues of IFMs are studied. Idempotent, regularity, permutation matrix and spectral radius of IFMs are also discussed. The parameterizations tool of IFM enhances the flexibility of its applications. For other works on IFMs see [1-3, 24, 30, 31, 34, 35].

The concept of IVFMs as a generalization of fuzzy matrix was introduced and developed in 2006 by Shaymal and Pal [36] by extending the max-min operation in fuzzy algebra. For more works on IVFMs see [28].

Combining IFMs and IVFMs, a new fuzzy matrix called interval-valued intuitionistic fuzzy matrices (IVIFMs) is defined [19]. For other works on IVIFMs, see [9, 11].

In [29], Pal has defined FMFRCs. In these matrices, rows and columns are also fuzzy numbers, i.e., unlike fuzzy matrices they are also uncertain. Motivated from this paper and interval-valued fuzzy graph, we define IVFMFRCs. The complement, some binary operations, density, etc. are defined and some important properties are proved.

2. Preliminaries

Some basic operations on interval-valued fuzzy numbers are defined below.

Let $\mathcal{D}$ denote the set of all subintervals of the interval $[0,1]$. Let $a = [a^-, a^+]$ and $b = [b^-, b^+]$ be two elements of $\mathcal{D}$. Then

1) $a \oplus b = [a^- + b^- - a^- \cdot b^-, a^+ + b^+ - a^+ \cdot b^+]$,

2) $a \odot b = [a^-, a^+] \odot [b^-, b^+] = [a^- \cdot b^-, a^+ \cdot b^+]$,

3) $a \wedge b = [a^-, a^+] \wedge [b^-, b^+] = [a^- \wedge b^-, a^+ \wedge b^+]$,

4) $a \vee b = [a^-, a^+] \vee [b^-, b^+] = [a^- \vee b^-, a^+ \vee b^+]$,

5) $[a@b = \left[\frac{a^- + b^-}{2}, \frac{a^+ + b^+}{2}\right]$,

6) $a^c = [1 - a^+, 1 - a^-]$.

The operators, "+", "−" and "·" used in extreme right are ordinary addition, subtraction and multiplication, respectively.

Two intervals $[a^-, a^+]$ and $[b^-, b^+]$ are equal if and only if $a^- = b^-$ and $a^+ = b^+$.

We denote $[0,0]$ and $[1,1]$ as $\mathbf{0}$ and $\mathbf{1}$ respectively.

Now we define interval-valued fuzzy matrix whose rows and columns are certain as follows.

Definition 2.1 *An interval-valued fuzzy matrix of order $m \times n$ is defined as, $A = (a_{ij})_{m\times n}$, where $a_{ij} = [a_{ij}^-, a_{ij}^+]$ is the ijth element of A, represents the membership value. All the elements of an IVFM are intervals and they are members of $\mathcal{D}$* [36].

In IVFM, the elements are the membership grade of some attributes, they are not crisp number, so naturally some new operations are needed to handle such matrices.

In IVFMs, it is assumed that all the rows and columns are certain, but in our new concept we assumed that the rows and columns are also uncertain and their membership values are members of $\mathcal{D}$. These type of matrices are defined in the next section.

3. New Definition of Interval-valued Fuzzy Matrices

Let $A = [r_A(i)][c_A(j)][a_{ij}]_{m\times n}$ be an IVFMFRC of order $m\times n$. Here $a_{ij}, i = 1, 2, \cdots, m$; $j = 1, 2, \cdots, n$ represents the ijth element of A, $r_A(i)$ and $c_A(j)$ represent the membership values of ith row and jth column respectively for $i = 1, 2, \cdots, m; j = 1, 2, \cdots, n$.

Let

$$A = \begin{array}{c} \\ r_A(1) \\ r_A(2) \\ \vdots \\ r_A(m) \end{array} \begin{array}{c} \begin{array}{cccc} c_A(1) & c_A(2) & \cdots & c_A(n) \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

be a matrix, where $r_A(i), i = 1, 2, \cdots, m$, $c_A(j), j = 1, 2, \cdots, n$ and $a_{ij}, i = 1, 2, \cdots, m$; $j = 1, 2, \cdots, n$, represent respectively the membership values of rows, columns and elements.

(i) When $r_A(i) = 1, i = 1, 2, \cdots, m$; $c_A(j) = 1, j = 1, 2, \cdots, n$ and $a_{ij} \in [0, 1]$, $i = 1, 2, \cdots, m$; $j = 1, 2, \cdots, n$, then A is an FM.

(ii) When $r_A(i) \in [0, 1], i = 1, 2, \cdots, m$; $c_A(j) \in [0, 1], j = 1, 2, \cdots, n$ and $a_{ij} \in [0, 1], i = 1, 2, \cdots, m$; $j = 1, 2, \cdots, n$, then A is called a fuzzy matrix with fuzzy rows and columns (FMFRC).

(iii) When $r_A(i) \in \mathcal{D}$, $i = 1, 2, \cdots, m$; $c_A(j) \in \mathcal{D}$, $j = 1, 2, \cdots, n$ and $a_{ij} \in \mathcal{D}$, $i = 1, 2, \cdots, m$; $j = 1, 2, \cdots, n$, then A is called an IVFMFRC. For this case, all $a_{ij} = [a^-_{ij}, a^+_{ij}], r_A(i) = [r^-_A(i), r^+_A(i)], c_A(i) = [c^-_A(i), c^+_A(i)]$ are members of $\mathcal{D}$.

The membership values of the rows of an IVFMFRC A may be written as an interval fuzzy vector $r_A = [r_A(1), r_A(2), \cdots, r_A(m)]$ and similarly $c_A = [c_A(1), c_A(2), \cdots, c_A(n)]$.

If the value of $r_A(i)$ or $c_A(j)$ is $\mathbf{0}$ for some i or j, then it implies that the ith row or jth column has no importance for the IVFMFRC A and in this case one can remove them from A. When $r_A(i) = \mathbf{1}$ and $c_A(j) = \mathbf{1}$ for all i, j, then IVFMFRC A becomes IVFM.

3.1. Equality of IVFMFRC

The equality of two IVFMFRCs can be defined in three different ways. Let $A = [r_A(i)][c_A(j)][a_{ij}]_{m\times n}$ and $B = [r_B(i)][c_B(j)][b_{ij}]_{m\times n}$ be two IVFMFRCs.

Type I: If $r_A(i) = r_B(i)$ and $c_A(j) = c_B(j)$ for $i = 1, 2, \cdots, m$; $j = 1, 2, \cdots, n$, whatever may be the relation between a_{ij} and b_{ij}, then we say that A and B are RC-equal and it is denoted by $A =_{RC} B$. If $r_A(i) \neq r_B(i)$ or $c_A(j) \neq c_B(j)$ for at least one i or j, then we say that $A \neq B$ in RC-equal sense. This is the weak equality between two IVFMFRCs and it occurs only in the IVFMFRCs and FMFRCs.

Type II: If $a_{ij} = b_{ij}$ for all i and j, whatever may be the values of $r_A(i)$, $r_B(i)$, $c_A(j)$, $c_B(j)$, then A and B are E-equal and it is denoted by $A =_E B$. This type of equality occurs in FMs also.

If $a_{ij} \neq b_{ij}$ for at least one i or j, then we say $A \neq_E B$ or $A \neq B$ in E-equal sense.

Type III: If both $A =_{RC} B$ and $A =_E B$, then we say that A and B are equal and it is denoted as $A = B$. That is, if

(i) $a_{ij} = b_{ij}$ for all i, j,

(ii) $r_A(i) = r_B(i)$ for all i,

(iii) $c_A(j) = c_B(j)$ for all j.

Then A is equal to B and is denoted as $A = B$. If A and B are not equal, then it is denoted by $A \neq B$. That is, if $A \neq_{RC} B$ and /or $A \neq_E B$, then we write $A \neq B$.

3.2. Null IVFMFRC

Based on the membership values of rows, columns and elements, three types of null IVFMFRC are defined.

Type I: If $r_A(i) = \mathbf{0}, c_A(j) = \mathbf{0}$ and $a_{ij} = \mathbf{0}$ for all i and j, then IVFMFRC A is called p-null, denoted by $\mathbf{0}_p$. For example

$$\begin{array}{c} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{array}\begin{array}{c} \mathbf{0}\ \mathbf{0}\ \mathbf{0} \\ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \end{array}$$

is a 3×3 order p-null IVFMFRC.

This concept is the same as FMs as well as classical matrices. This type of null matrix is called empty IVFMFRC.

Type II: If $a_{ij} = \mathbf{0}$ for all i and j, whatever may be the values of $r_A(i)$ and $c_A(j)$, then the IVFMFRC A is called E-null, and it is denoted by $\mathbf{0}_E$. For example

$$\begin{array}{c} \\ [0.2, 0.4] \\ [0.1, 0.5] \\ [0.2, 0.8] \end{array}\begin{array}{c} [0.5, 0.6]\ [0.0, 0.1]\ [0.6, 0.7] \\ \begin{bmatrix} [0.0, 0.0] & [0.0, 0.0] & [0.0, 0.0] \\ [0.0, 0.0] & [0.0, 0.0] & [0.0, 0.0] \\ [0.0, 0.0] & [0.0, 0.0] & [0.0, 0.0] \end{bmatrix} \end{array}$$

is a 3×3 order E-null IVFMFRC.

This concept is also similar to FMs.

Type III: If $r_A(i) = \mathbf{0}, c_A(j) = \mathbf{0}$ for all i and j, whatever may be the values of a_{ij}, then the IVFMFRC is called RC-null and it is denoted by $\mathbf{0}_{RC}$.

For example

$$\begin{array}{c} \\ [0.0, 0.0] \\ [0.0, 0.0] \\ [0.0, 0.0] \end{array}\begin{array}{c} [0.0, 0.0]\ [0.0, 0.0]\ [0.0, 0.0]\ \ [0.0, 0.0] \\ \begin{bmatrix} [0.2, 0.5] & [0.0, 0.1] & [0.3, 0.5] & [0.10, 0.15] \\ [0.3, 0.8] & [0.6, 0.8] & [0.1, 0.9] & [0.45, 0.85] \\ [0.2, 0.5] & [0.2, 0.7] & [0.5, 0.5] & [0.56, 0.71] \end{bmatrix} \end{array}$$

is a RC-null IVFMFRC.

This type of null matrix is new and it is only defined for IVFMFRC and FMFRC.

3.3. Identity IVFMFRC

Two types of identity IVFMFRC are defined here.

Type I: A square IVFMFRC of order $n \times n$ is called p-identity IVFMFRC if $r_A(i) = \mathbf{1}$ and $c_A(j) = \mathbf{1}$ for all i and j and $a_{ii} = \mathbf{1}$, $a_{ij} = \mathbf{0}, i \neq j$ for all i, j. It is denoted by I_p. For example

$$\begin{array}{c|c} & \mathbf{1}\ \mathbf{1}\ \mathbf{1} \\ \begin{array}{c}\mathbf{1}\\ \mathbf{1}\\ \mathbf{1}\end{array} & \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \end{array}$$

is a 3×3 order p-identity IVFMFRC.

Type II: A square IVFMFRC of order $n \times n$ is called f-identity if $a_{ii} = \mathbf{1}, a_{ij} = \mathbf{0}, i \neq j$ for all i, j, whatever may be the values of $r_A(i)$ and $c_A(j)$ and it is denoted by I_f. For example

$$\begin{array}{c c} & [0.2, 0.5]\ [0.3, 0.7]\ [0.4, 0.8] \\ \begin{array}{c}[0.2, 0.7]\\ [0.5, 0.7]\\ [0.1, 0.4]\end{array} & \begin{bmatrix} [1.0, 1.0] & [0.0, 0.0] & [0.0, 0.0] \\ [0.0, 0.0] & [1.0, 1.0] & [0.0, 0.0] \\ [0.0, 0.0] & [0.0, 0.0] & [1.0, 1.0] \end{bmatrix} \end{array}$$

is a 3×3 order f-identity IVFMFRC.

4. Operators on IVFMFRCs

In this section, some operators, viz. $\vee, \wedge, \odot, \oplus$ are defined and explained with numerical examples.

4.1. $\vee$ Operator

Let $A = [r_A(i)][c_A(j)][a_{ij}]_{m\times n}$ and $B = [r_B(i)][c_B(j)][b_{ij}]_{m\times n}$ be two IVFMFRCs. Then

$$A \vee B = D = [r_D(i)][c_D(j)][d_{ij}]_{m\times n},$$

where

$r_D(i) = r_A(i) \vee r_B(i) = [r_A^-(i) \vee r_B^-(i),\ r_A^+(i) \vee r_B^+(i)]$,
$c_D(j) = c_A(j) \vee c_B(j) = [c_A^-(j) \vee c_B^-(j),\ c_A^+(j) \vee c_B^+(j)]$ and
$d_{ij} = a_{ij} \vee b_{ij} = [a_{ij}^- \vee b_{ij}^-,\ a_{ij}^+ \vee b_{ij}^+]$ for all i, j.

Note that the order of A and B must be equal.

But, in our new concept one can operate two IVFMFRCs with different orders. Suppose $A = [r_A(i)][c_A(j)][a_{ij}]_{m\times n}$ and $B = [r_B(i)][c_B(j)][b_{ij}]_{p\times q}$ be two IVFMFRCs. For the sack of simplicity, we assume that $m \leq p$ and $n \leq q$. If $m = p$ and $n = q$, then there is nothing new.

Otherwise, three different cases may arise:

(i) $m < p, \quad n \leq q$,

(ii) $m \leq p, \quad n < q$,

(iii) $m < p, \quad n < q$.

In these cases, add $p - m$ (may be 0) rows and $q - n$ (may also be 0) columns at the end of rows and columns.

The elements of these rows and columns are taken as $\mathbf{0}$ and membership values of all rows and columns are taken as $\mathbf{0}$. After introduction of these rows and columns, the IVFMFRC A becomes an IVFMFRC of order $p\times q$. Now, the $\vee$ operation can be performed as in previous case. Note that the order of $A\vee B$ becomes $p\times q$.

To illustrate this new concept, let us consider two IVFMFRCs as

$$A=\begin{array}{c c} & \begin{array}{ccc}[0.1,0.7] & [0.7,1.0] & [0.6,0.8]\end{array} \\ \begin{array}{c}[0.1,0.5]\\ [0.4,0.6]\end{array} & \begin{bmatrix}[0.0,0.1] & [0.4,0.8] & [0.3,0.7]\\ [0.2,0.5] & [0.6,0.7] & [0.0,1.0]\end{bmatrix}\end{array}$$

and

$$B=\begin{array}{c c} & \begin{array}{cc}[0.2,0.5] & [0.1,0.6]\end{array} \\ \begin{array}{c}[0.0,0.2]\\ [0.6,0.8]\\ [0.3,0.9]\end{array} & \begin{bmatrix}[0.0,0.1] & [0.0,0.2]\\ [0.6,0.8] & [0.3,0.9]\\ [0.0,0.3] & [0.1,0.7]\end{bmatrix}\end{array}.$$

Note that the number of rows of A is less than that of B and the number of columns of B is less than that of A. The augmented matrices A_a and B_a are given by

$$A_a=\begin{array}{c c} & \begin{array}{ccc}[0.1,0.7] & [0.7,1.0] & [0.6,0.8]\end{array} \\ \begin{array}{c}[0.1,0.5]\\ [0.4,0.6]\\ [0.0,0.0]\end{array} & \begin{bmatrix}[0.0,0.1] & [0.4,0.8] & [0.3,0.7]\\ [0.2,0.5] & [0.6,0.7] & [0.0,1.0]\\ [0.0,0.0] & [0.0,0.0] & [0.0,0.0]\end{bmatrix}\end{array}$$

and

$$B_a=\begin{array}{c c} & \begin{array}{ccc}[0.2,0.5] & [0.1,0.6] & [0.0,0.0]\end{array} \\ \begin{array}{c}[0.0,0.2]\\ [0.6,0.8]\\ [0.3,0.9]\end{array} & \begin{bmatrix}[0.0,0.1] & [0.0,0.2] & [0.0,0.0]\\ [0.6,0.8] & [0.3,0.9] & [0.0,0.0]\\ [0.0,0.3] & [0.1,0.7] & [0.0,0.0]\end{bmatrix}\end{array}.$$

Therefore, $A\vee B=A_a\vee B_a=\begin{array}{c c} & \begin{array}{ccc}[0.2,0.7] & [0.7,1.0] & [0.6,0.8]\end{array} \\ \begin{array}{c}[0.1,0.5]\\ [0.6,0.8]\\ [0.3,0.9]\end{array} & \begin{bmatrix}[0.0,0.1] & [0.4,0.8] & [0.3,0.7]\\ [0.6,0.8] & [0.6,0.9] & [0.0,1.0]\\ [0.0,0.3] & [0.1,0.7] & [0.0,0.0]\end{bmatrix}\end{array}.$

Note that the order of A and B are 2×3 and 3×2, but the order of $A\vee B$ is 3×3.

Observations: The following results are obvious:

(i) $\mathbf{0}_p\vee A=A$;

(ii) (a) $\mathbf{0}_E\vee A\neq A$,

(b) $\mathbf{0}_E\vee A\neq_{RC} A$,

(c) $\mathbf{0}_E\vee A=_E A$;

(iii) (a) $\mathbf{0}_{RC}\vee A\neq A$,

(b) $\mathbf{0}_{RC}\vee A\neq_{RC} A$,

(c) $\mathbf{0}_{RC}\vee A=_E A$.

4.2. $\wedge$ Operator

The $\wedge$ operation is similar to $\vee$ operation. Let $A = [r_A(i)][c_A(j)][a_{ij}]_{m\times n}$ and $B = [r_B(i)][c_B(j)][b_{ij}]_{m\times n}$ be two IVFMFRCs. Then

$$A \wedge B = D = [r_D(i)][c_D(j)][d_{ij}]_{m\times n},$$

where

$r_D(i) = r_A(i) \wedge r_B(i) = [r_A^-(i) \wedge r_B^-(i),\ r_A^+(i) \wedge r_B^+(i)],$
$c_D(j) = c_A(j) \wedge c_B(j) = [c_A^-(i) \wedge c_B^-(i),\ c_A^+(i) \wedge c_B^+(i)]$

and

$d_{ij} = a_{ij} \wedge b_{ij} = [a_{ij}^- \wedge b_{ij}^-,\ a_{ij}^+ \wedge b_{ij}^+]$ for all i, j.

If the orders of A and B are different, then this case can be handled as in case of $\vee$ operation.

4.3. $\oplus, \odot$ and @ Operators

Let $A = [r_A(i)][c_A(j)][a_{ij}]_{m\times n}$ and $B = [r_B(i)][c_B(j)][b_{ij}]_{m\times n}$ be two IVFMFRCs. Then

$$A \oplus B = D = [r_D(i)][c_D(j)][d_{ij}]_{m\times n},$$

where

$r_D(i) = r_A(i) \oplus r_B(i) = [r_A^-(i) + r_B^-(i) - r_A^-(i) \cdot r_B^-(i),\ r_A^+(i) + r_B^+(i) - r_A^+(i) \cdot r_B^+(i)],$
$c_D(j) = c_A(j) \oplus c_B(j) = [c_A^-(j) + c_B^-(j) - c_A^-(j) \cdot c_B^-(j),\ c_A^+(j) + c_B^+(j) - c_A^+(j) \cdot c_B^+(j)],$

and

$d_{ij} = a_{ij} \oplus b_{ij} = [a_{ij}^- + b_{ij}^- - a_{ij}^- \cdot b_{ij}^-,\ a_{ij}^+ + b_{ij}^+ - a_{ij}^+ \cdot b_{ij}^+]$ for all i and j.

The operator $\odot$ is defined as

$$A \odot B = E = [r_E(i)][c_E(j)][e_{ij}]_{m\times n},$$

$r_E(i) = r_A(i) \odot r_B(i) = [r_A^-(i) \cdot r_B^-(i),\ r_A^+(i) \cdot r_B^+(i)],$
$c_E(j) = c_A(j) \odot c_B(j) = [c_A^-(j) \cdot c_B^-(j),\ c_A^+(j) \cdot c_B^+(j)]$, and
$e_{ij} = a_{ij} \odot b_{ij} = [a_{ij}^- \cdot b_{ij}^-,\ a_{ij}^+ \cdot b_{ij}^+]$ for all i and j

and

$$A@B = F = [r_F(i)][c_F(j)][f_{ij}]_{m\times n},$$

$r_F(i) = r_A(i)@r_B(i) = [\frac{1}{2}(r_A^-(i) + r_B^-(i)),\ \frac{1}{2}(r_A^+(i) + r_B^+(i))],$
$c_F(j) = c_A(j)@c_B(j) = [\frac{1}{2}(c_A^-(j) + c_B^-(j)),\ \frac{1}{2}(c_A^+(j) + c_B^+(j))]$, and
$f_{ij} = a_{ij}@b_{ij} = [\frac{1}{2}(a_{ij}^- + b_{ij}^-),\ \frac{1}{2}(a_{ij}^+ + b_{ij}^+)]$ for all i and j.

5. g-IVFMFRC

In this section, a special type of IVFMFRC is defined along with other two types of IVFMFRCs.

Definition 5.1 *If $a_{ij}^- \leq r_A^-(i) \wedge c_A^-(j)$ and $a_{ij}^+ \leq r_A^+(i) \wedge c_A^+(j)$ for all i, j, then the IVFMFRC $A = [r_A(i)][c_A(j)][a_{ij}]_{m\times n}$ is called g-IVFMFRC.*

Let

$$A = \begin{array}{c} \\ [0.1, 0.5] \\ [0.2, 0.8] \\ [0.6, 0.7] \end{array} \begin{array}{c} \begin{array}{ccc} [0.1, 0.2] & [0.2, 1.0] & [0.5, 0.8] \end{array} \\ \begin{bmatrix} [0.0, 0.2] & [0.1, 0.4] & [0.0, 0.3] \\ [0.1, 0.2] & [0.2, 0.8] & [0.5, 0.7] \\ [0.0, 0.1] & [0.1, 0.6] & [0.4, 0.6] \end{bmatrix} \end{array}$$

and

$$B = \begin{array}{c} \\ [0.2, 0.7] \\ [0.3, 0.8] \end{array} \begin{array}{c} \begin{array}{ccc} [0.5, 0.7] & [0.1, 0.6] & [0.4, 1.0] \end{array} \\ \begin{bmatrix} [0.2, 0.6] & [0.1, 0.6] & [0.0, 0.7] \\ [0.3, 0.5] & [0.1, 0.5] & [0.3, 0.7] \end{bmatrix} \end{array}$$

both are g-IVFMFRCs.

Definition 5.2 *If $a^-_{ij} = r^-_A(i) \wedge c^-_A(j)$ and $a^+_{ij} = r^+_A(i) \wedge c^+_A(j)$ for all i, j, then the IVFMFRC A is called a complete IVFMFRC.*

From the definition it is obvious that every complete IVFMFRC is g-IVFMFRC, but converse is not true.

The IVFMFRC

$$A = \begin{array}{c} \\ [0.1, 0.6] \\ [0.3, 0.4] \end{array} \begin{array}{c} \begin{array}{ccc} [0.2, 0.5] & [0.3, 0.7] & [0.5, 0.9] \end{array} \\ \begin{bmatrix} [0.1, 0.5] & [0.1, 0.6] & [0.1, 0.6] \\ [0.2, 0.4] & [0.3, 0.4] & [0.3, 0.4] \end{bmatrix} \end{array}$$

is a complete IVFMFRC.

Now we define another kind of IVFMFRC.

Definition 5.3 *If $a^-_{ij} \leq r^-_A(i) \bullet c^-_A(j)$ and $a^+_{ij} \leq r^+_A(i) \bullet c^+_A(j)$ for all i, j, then the IVFMFRC A is called a dot IVFMFRC, where $\bullet$ denotes ordinary multiplication.*

The IVFMFRC is

$$A = \begin{array}{c} \\ [0.4, 0.8] \\ [0.2, 0.7] \\ [0.6, 0.7] \end{array} \begin{array}{c} \begin{array}{ccc} [0.6, 0.9] & [0.7, 1.0] & [0.2, 0.5] \end{array} \\ \begin{bmatrix} [0.2, 0.7] & [0.2, 0.7] & [0.0, 0.4] \\ [0.1, 0.5] & [0.1, 0.6] & [0.0, 0.2] \\ [0.3, 0.6] & [0.4, 0.5] & [0.1, 0.3] \end{bmatrix} \end{array}$$

Lemma 5.1 *Every dot IVFMFRC is a g-IVFMFRC.*

Proof Since $0 \leq r^-_A(i) \leq 1$, $0 \leq r^+_A(i) \leq 1$ and $0 \leq c^-_A(i) \leq 1$, $0 \leq c^+_A(i) \leq 1$ $r^-_A(i) \bullet c^-_A(j) \leq r^-_A(i) \wedge c^-_A(j)$ and $r^+_A(i) \bullet c^+_A(j) \leq r^+_A(i) \wedge c^+_A(j)$ for all i, j.

If A is a dot IVFMFRC, then $a^-_{ij} \leq r^-_A(i) \bullet c^-_A(j)$ and $a^+_{ij} \leq r^+_A(i) \bullet c^+_A(j)$ for all i, j. Therefore, for all i, j, $a^-_{ij} \leq r^-_A(i) \bullet c^-_A(j) \leq r^-_A(i) \wedge c^-_A(j)$ and $a^+_{ij} \leq r^+_A(i) \bullet c^+_A(j) \leq r^+_A(i) \wedge c^+_A(j)$, i.e. $a^-_{ij} \leq r^-_A(i) \wedge c^-_A(j)$ and $a^+_{ij} \leq r^+_A(i) \wedge c^+_A(j)$ for all i, j.

Hence, A is a g-IVFMFRC.

6. Complement of IVFMFRC

In this section, two types of complements of an IVFMFRC are defined. Like fuzzy matrix, the complement of an IVFMFRC is defined below:

We define $\mathbf{1}-a, a \in \mathcal{D}$ as follows:

$$\mathbf{1}-a=[1,1]-[a^-,a^+]=[1-a^+,1-a^-].$$

Note that $0 \le 1-a^+ \le 1-a^- \le 1$ and hence $\mathbf{1}-a \in \mathcal{D}$.

Definition 6.1 *Let $A=[r_A(i)][c_A(j)][a_{ij}]_{m\times n}$ be an IVFMFRC. Its complement is denoted by A^c and it is defined as $A^c=[\mathbf{1}-r_A(i)][\mathbf{1}-c_A(j)][\mathbf{1}-a_{ij}]_{m\times n}$.*

Definition 6.2 *An IVFMFRC is called self complement if $A^c=A$.*

Theorem 6.1 *If A is an IVFMFRC, then $(A^c)^c=A$.*

Proof Let $B=A^c$. Then $r_B(i)=\mathbf{1}-r_A(i), c_B(j)=\mathbf{1}-c_A(j), b_{ij}=\mathbf{1}-a_{ij}=[1-a^+,1-a^-]$. If $D=B^c=(A^c)^c$, then $d_{ij}=\mathbf{1}-b_{ij}=[1-b^+_{ij},1-b^-_{ij}]=[1-(1-a^-_{ij}),1-(1-a^+_{ij})]=[a^-_{ij},a^+_{ij}]=a_{ij}$.

Similarly, $r_D(i)=r_A(i), c_D(j)=c_A(j)$ for all i,j.

Hence, $D=A$, i.e. $(A^c)^c=A$.

Theorem 6.2 *If an IVFMFRC $A=[r_A(i)][c_A(j)][a_{ij}]_{m\times n}$ is self complement, then $r^-_A(i)+r^+_A(i)=1, c^-_A(i)+c^+_A(i)=1$, $a^-_{ij}+a^+_{ij}=1$, for all i,j.*

Proof By the definition of complement, $a^c_{ij}=[1,1]-[a^-_{ij},a^+_{ij}]=[1-a^+_{ij},1-a^-_{ij}]$. Since $A^c=A$, $[1-a^+_{ij},1-a^-_{ij}]=[a^-_{ij},a^+_{ij}]$. This gives $a^-_{ij}+a^+_{ij}=1$ for all i,j.

The other parts of the proof are similar.

Theorem 6.3 (De Morgan's laws) *Let $A=[r_A(i)][c_A(j)][a_{ij}]_{m\times n}$ and $B=[r_B(i)][c_B(j)]$ $[b_{ij}]_{m\times n}$ be two IVFMFRCs. Then*

(i) $(A\vee B)^c=A^c\wedge B^c$,

(ii) $(A\wedge B)^c=A^c\vee B^c$.

Proof (i) Let $D=A\vee B$. Then $r_D(i)=r_A(i)\vee r_B(i), c_D(j)=c_A(j)\vee c_B(j), d_{ij}=a_{ij}\vee b_{ij}$.

Let $E=D^c$. Then $r_E(i)=\mathbf{1}-r_D(i)=\mathbf{1}-r_A(i)\vee r_B(i), c_E(j)=\mathbf{1}-c_D(j)=\mathbf{1}-c_A(j)\vee c_B(j), e_{ij}=\mathbf{1}-d_{ij}=\mathbf{1}-a_{ij}\vee b_{ij}=[1,1]-[a^-_{ij}\vee b^-_{ij},a^+_{ij}\vee b^+_{ij}]=[1-a^+_{ij}\vee b^+_{ij},1-a^-_{ij}\vee b^-_{ij}]$. Let $F=A^c\wedge B^c$.

Therefore, $f_{ij}=(\mathbf{1}-a_{ij})\wedge(\mathbf{1}-b_{ij})=([1,1]-[a^-_{ij},a^+_{ij}])\wedge([1,1]-[b^-_{ij},b^+_{ij}])=[1-a^+_{ij},1-a^-_{ij}]\wedge[1-b^+_{ij},1-b^-_{ij}]=[(1-a^+_{ij})\wedge(1-b^+_{ij}),(1-a^-_{ij})\wedge(1-b^-_{ij})]=[1-a^+_{ij}\vee b^+_{ij},1-a^-_{ij}\vee b^-_{ij}]=e_{ij}$.

Similarly, $r_F(i)=r_E(i), c_F(j)=c_E(j)$ for all i,j.

Hence, $(A\vee B)^c=A^c\wedge B^c$.

(ii) Proof is similar to (i).

$$A=\begin{array}{c}\\ [0.1,0.7]\\ [0.2,0.5]\end{array}\begin{array}{c}\begin{array}{ccc}[0.3,0.6] & [0.1,0.8] & [0.2,0.7]\end{array}\\ \begin{bmatrix}[0.0,0.2] & [0.4,0.5] & [0.5,0.7]\\ [0.7,0.9] & [0.3,0.8] & [0.3,0.6]\end{bmatrix}\end{array}.$$

and

$$B = \begin{array}{cc} & \begin{array}{cc} [0.0, 0.2] & [0.3, 0.5] \end{array} \\ \begin{array}{c} [0.2, 0.7] \\ [0.5, 0.9] \\ [0.2, 0.4] \end{array} & \begin{bmatrix} [0.3, 0.4] & [0.3, 0.5] \\ [0.0, 0.3] & [0.5, 0.7] \\ [0.3, 0.5] & [0.8, 1.0] \end{bmatrix} \end{array}.$$

Before going to such operations we augment A and B to make same size and let the augmented matrices be A_a and B_a corresponding to A and B.

Then

$$A_a \vee B_a = \begin{array}{cc} & \begin{array}{ccc} [0.3, 0.6] & [0.3, 0.8] & [0.2, 0.7] \end{array} \\ \begin{array}{c} [0.2, 0.7] \\ [0.5, 0.9] \\ [0.2, 0.4] \end{array} & \begin{bmatrix} [0.3, 0.4] & [0.4, 0.5] & [0.5, 0.7] \\ [0.7, 0.9] & [0.5, 0.8] & [0.3, 0.6] \\ [0.3, 0.5] & [0.8, 1.0] & [0.0, 0.0] \end{bmatrix} \end{array}.$$

Now,

$$(A_a \vee B_a)^c = \begin{array}{cc} & \begin{array}{ccc} [0.4, 0.7] & [0.2, 0.7] & [0.3, 0.8] \end{array} \\ \begin{array}{c} [0.3, 0.8] \\ [0.1, 0.5] \\ [0.6, 0.8] \end{array} & \begin{bmatrix} [0.6, 0.7] & [0.5, 0.6] & [0.3, 0.5] \\ [0.1, 0.3] & [0.2, 0.5] & [0.4, 0.7] \\ [0.5, 0.7] & [0.0, 0.2] & [1.0, 1.0] \end{bmatrix} \end{array}.$$

Again,

$$A_a^c = \begin{array}{cc} & \begin{array}{ccc} [0.4, 0.7] & [0.2, 0.9] & [0.3, 0.8] \end{array} \\ \begin{array}{c} [0.3, 0.9] \\ [0.5, 0.8] \\ [1.0, 1.0] \end{array} & \begin{bmatrix} [0.8, 1.0] & [0.5, 0.6] & [0.3, 0.5] \\ [0.1, 0.3] & [0.2, 0.7] & [0.4, 0.7] \\ [1.0, 1.0] & [1.0, 1.0] & [1.0, 1.0] \end{bmatrix} \end{array}$$

and

$$B_a^c = \begin{array}{cc} & \begin{array}{ccc} [0.8, 1.0] & [0.5, 0.7] & [1.0, 1.0] \end{array} \\ \begin{array}{c} [0.3, 0.8] \\ [0.1, 0.5] \\ [0.6, 0.8] \end{array} & \begin{bmatrix} [0.6, 0.7] & [0.5, 0.7] & [1.0, 1.0] \\ [0.7, 1.0] & [0.3, 0.5] & [1.0, 1.0] \\ [0.5, 0.7] & [0.0, 0.2] & [1.0, 1.0] \end{bmatrix} \end{array}.$$

Now,

$$A_a^c \wedge B_a^c = \begin{array}{cc} & \begin{array}{ccc} [0.4, 0.7] & [0.2, 0.7] & [0.3, 0.8] \end{array} \\ \begin{array}{c} [0.3, 0.8] \\ [0.1, 0.5] \\ [0.6, 0.8] \end{array} & \begin{bmatrix} [0.6, 0.7] & [0.5, 0.6] & [0.3, 0.5] \\ [0.1, 0.3] & [0.2, 0.5] & [0.4, 0.7] \\ [0.5, 0.7] & [0.0, 0.2] & [1.0, 1.0] \end{bmatrix} \end{array}.$$

Hence, $(A_a \vee B_a)^c = A_a^c \wedge B_a^c$.

The above result is true for all IVFMFRCs, if both IVFMFRCs have same size. But, if their sizes are different, then the matrices should be augmented to make same size before computing complement.

Let us define a new operator $\diamond$ for the IVFMFRCs $A = [r_A(i)][c_A(j)][a_{ij}]_{m\times n}$ and $B = [r_B(i)][c_B(j)][b_{ij}]_{m\times n}$ as

$$A \diamond B = (A^c \wedge B) \vee (A \wedge B^c).$$

Theorem 6.4 *For any IVFMFRCs A and B,* $A^c \diamond B^c = A \diamond B$.

Proof By definition

$$A^c \diamond B^c = \{(A^c)^c \wedge B^c\} \vee \{A^c \wedge (B^c)^c\} = (A \wedge B^c) \vee (A^c \wedge B) = A \diamond B.$$

7. Density of IVFMFRC

Sometimes we see that a matrix, it may be classical matrix or fuzzy matrix, contains many zeros. This type of matrix is called sparse matrix and special methods are used to store such matrix in computer. In contrast, if the number of non-zero elements is high then the matrix is called dense matrix.

The density of an FM is defined below.

Definition 7.1 *Let* $A = [a_{ij}]_{m\times n}$ *be an FM. The density of A is denoted as* $D(A)$ *and is defined as*

$$D(A) = \frac{1}{mn}\sum_{i,j} a_{ij}.$$

From definition it follows that $0 \le D(A) \le 1$ for an FM A. Actually, $D(A)$ represents the average membership of the elements in the FM A.

But, in FMFRC, rows and columns are not certain, and hence the density is to be re-defined for FMFRC. The definition is given below.

Definition 7.2 [29] *Let* $A = [r_A(i)][c_A(j)][a_{ij}]_{m\times n}$ *be an FMFRC. The density of A is denoted by* $D(A)$ *and is defined as*

$$D(A) = \frac{\sum\limits_{i,j} a_{ij}}{\sum\limits_{i,j} r_A(i) \wedge c_A(j)},$$

provided $\sum\limits_{i,j} r_A(i) \wedge c_A(j) \neq 0$.

For an IVFMFRC, $a_{ij} \geq 0$ for all i, j. Thus, $D(A)$ is non-negative for any IVFMFRC A. $D(A)$ is zero only when all $a_{ij} = 0$. Higher value of $D(A)$, indicates the matrix is more dense. If the value of $D(A)$ is closed to zero, then the IVFMFRC is called sparse IVFMFRC with degree of sparsity $D(A)$.

Theorem 7.1 [29] *For a g-FMFRC,* $0 \le D(A) \le 1$.

Theorem 7.2 [29] *If A is a complete g-IVFMFRC, then* $D(A) = 1$.

Like classical sub-matrix, one can defined sub-IVFMFRC. Any portion, not necessarily consecutive, along with corresponding rows and columns membership values is called sub-IVFMFRC.

Like FMFRC, the density of an IVFMFRC is defined as $D(A) = [D^-(A), D^+(A)]$, where

$$D^-(A) = \frac{\sum_{i,j} a^-_{ij}}{\sum_{i,j} r^+_A(i) \wedge c^+_A(j)} \text{ and } D^+(A) = \frac{\sum_{i,j} a^+_{ij}}{\sum_{i,j} r^+_A(i) \wedge c^+_A(j)}.$$

Since $a^-_{ij} \le a^+_{ij}$ for all i, j, therefore it is easy to verify that $D^-(A) \le D^+(A)$.

Let

$$A = \begin{array}{cc} & \begin{array}{cc} [0.6, 0.9] & [0.0, 1.0] \end{array} \\ \begin{array}{c} [0.2, 0.5] \\ [0.3, 0.8] \end{array} & \begin{bmatrix} [0.5, 0.6] & [0.8, 1.0] \\ [0.3, 0.7] & [0.6, 0.8] \end{bmatrix} \end{array}$$

and

$$B = \begin{array}{cc} & \begin{array}{cc} [0.6, 0.9] & [0.5, 1.0] \end{array} \\ \begin{array}{c} [0.6, 0.8] \\ [0.7, 0.9] \end{array} & \begin{bmatrix} [0.5, 0.6] & [0.5, 0.7] \\ [0.6, 0.9] & [0.4, 0.8] \end{bmatrix} \end{array}.$$

Then

$$D^-(A) = \frac{0.5 + 0.8 + 0.3 + 0.6}{0.5 + 0.5 + 0.8 + 0.8} = \frac{11}{13} \text{ and } D^+(A) = \frac{0.6 + 1.0 + 0.7 + 0.8}{0.5 + 0.5 + 0.8 + 0.8} = \frac{31}{26}.$$

Thus density of A, $D(A) = [11/13, 31/26]$.

Also,

$$D^-(B) = \frac{0.5 + 0.5 + 0.6 + 0.4}{0.8 + 0.8 + 0.9 + 0.9} = \frac{10}{17} \text{ and } D^+(B) = \frac{0.6 + 0.7 + 0.9 + 0.8}{0.8 + 0.8 + 0.9 + 0.9} = \frac{15}{17}.$$

Therefore, $D(B) = \left[\frac{10}{17}, \frac{15}{17}\right]$.

Note that B is a g-IVFMFRC, and in this case the density belongs to $\mathcal{D}$. In general, there is no upper bound of $D(A)$ if A is any IVFMFRC, but it is bounded when it is g-IVFMFRC.

Theorem 7.3 *If A is a g-IVFMFRC, then $D(A) \in \mathcal{D}$.*

Proof It is obvious that $D^-(A) \le D^+(A)$. We shall prove that $D^+(A) \le 1$.

Since A is a g-IVFMFRC, $a^+_{ij} \le r^+_A(i) \wedge c^+_A(j)$ for all i, j. Therefore,

$$\sum_{i,j} a^+_{ij} \le \sum_{i,j} r^+_A(i) \wedge c^+_A(j).$$

That is,

$$\frac{\sum_{i,j} a^+_{ij}}{\sum_{i,j} r^+_A(i) \wedge c^+_A(j)} \le 1.$$

Hence $D^+(A) \le 1$. Thus, $D(A) \in \mathcal{D}$.

Definition 7.3 *An IVFMFRC A is called balanced if $D^-(S) \le D^-(A)$ and $D^+(S) \le D^+(A)$ for all sub-IVFMFRCs S of A.*

Let $A = \begin{array}{cc} & \begin{array}{cc} [0.8, 1.0] & [0.5, 0.6] \end{array} \\ \begin{array}{c} [0.6, 0.9] \\ [0.7, 1.0] \end{array} & \begin{bmatrix} [0.5, 0.5] & [0.5, 0.8] \\ [0.7, 0.9] & [0.3, 0.4] \end{bmatrix} \end{array}$.

Let

$$S_1 = \begin{array}{cc} & [0.8, 1.0] \\ [0.6, 0.9] & \begin{bmatrix} [0.5, 0.5] \end{bmatrix} \end{array},$$

$$S_2 = \begin{array}{cc} & [0.5, 0.6] \\ [0.6, 0.9] & \left[\begin{array}{c} [0.5, 0.8] \end{array}\right] \end{array},$$

$$S_3 = \begin{array}{cc} & [0.8, 1.0] \\ [0.7, 1.0] & \left[\begin{array}{c} [0.7, 0.9] \end{array}\right] \end{array},$$

$$S_4 = \begin{array}{cc} & [0.5, 0.6] \\ [0.7, 1.0] & \left[\begin{array}{c} [0.3, 0.4] \end{array}\right] \end{array},$$

be the sub-IVFMFRCs of A.

Now,

$D(S_1) = [5/9, 5/9]$, $D(S_2) = [5/6, 4/3]$, $D(S_3) = [7/10, 9/10]$,
$D(S_4) = [1/2, 2/3]$, $D(A) = [20/31, 26/31]$.

Note that $D^-(S_2) > D^-(A)$, that is, A is not a balanced IVFMFRC.

Now, we define a particular type of balanced IVFMFRC.

Definition 7.4 *An IVFMFRC A is called strictly balanced if* $D^-(S) = D^-(A)$ *and* $D^+(S) = D^+(A)$ *for all sub-IVFMFRCs S of A.*

$$\text{Let } B = \begin{array}{ccc} & [0.4, 0.8] & [0.2, 0.5] \\ \begin{array}{c} [0.2, 0.6] \\ [0.5, 0.7] \end{array} & \multicolumn{2}{c}{\left[\begin{array}{cc} [3/25, 3/5] & [1/10, 1/2] \\ [7/50, 7/10] & [1/10, 1/2] \end{array}\right]} \end{array}.$$

Let

$$S_1 = \begin{array}{cc} & [0.4, 0.8] \\ [0.2, 0.6] & \left[\begin{array}{c} [3/25, 3/5] \end{array}\right] \end{array}, \qquad S_2 = \begin{array}{cc} & [0.2, 0.5] \\ [0.2, 0.6] & \left[\begin{array}{c} [1/10, 1/2] \end{array}\right] \end{array},$$

$$S_3 = \begin{array}{cc} & [0.4, 0.8] \\ [0.5, 0.7] & \left[\begin{array}{c} [7/50, 7/10] \end{array}\right] \end{array}, \qquad S_4 = \begin{array}{cc} & [0.2, 0.5] \\ [0.5, 0.7] & \left[\begin{array}{c} [1/10, 1/2] \end{array}\right] \end{array}$$

be the sub-IVFMFRCs of B.

Now, $D(S_1) = [1/5, 1]$, $D(S_2) = [1/5, 1]$, $D(S_3) = [1/5, 1]$, $D(S_4) = [1/5, 1]$, $D(B) = [1/5, 1]$.

Since $D(S_1) = D(S_2) = D(S_3) = D(S_4) = D(S) = [1/5, 1]$, therefore B is a strictly balanced IVFMFRC.

Theorem 7.4 *If A is a complete g-IVFMFRC, then* $D^-(A) \leq D^+(A)$ *and* $D^+(A) = 1$.

Proof Let A be a complete g-IVFMFRC. Therefore, $a_{ij} = r_A(i) \wedge c_A(j)$, i.e., $a^-_{ij} = r^-_A(i) \wedge c^-_A(j)$ and $a^+_{ij} = r^+_A(i) \wedge c^+_A(j)$ for all i, j.

Now,

$$D^+(A) = \frac{\sum_{i,j} a^+_{ij}}{\sum_{i,j} r^+_A(i) \wedge c^+_A(j)} \text{ and } D^-(A) = \frac{\sum_{i,j} a^-_{ij}}{\sum_{i,j} r^+_A(i) \wedge c^+_A(j)}.$$

Thus, $D^+(A) = 1$ and obviously, $D^-(A) \leq D^+(A)$.

$$\text{Let } A = \begin{array}{ccc} & [0.2, 0.5] & [0.4, 0.6] \\ \begin{array}{c} [0.3, 0.7] \\ [0.1, 0.7] \end{array} & \multicolumn{2}{c}{\left[\begin{array}{cc} [0.2, 0.5] & [0.3, 0.6] \\ [0.1, 0.5] & [0.1, 0.6] \end{array}\right]} \end{array}.$$

Let

$$S_1 = \begin{array}{cc} & [0.2, 0.5] \\ [0.3, 0.7] & \left[\begin{array}{c} [0.2, 0.5] \end{array}\right] \end{array},$$

$$S_2 = \begin{array}{cc} & [0.4, 0.6] \\ [0.3, 0.7] & \big[\ [0.3, 0.6]\ \big] \end{array},$$

$$S_3 = \begin{array}{cc} & [0.2, 0.5] \\ [0.1, 0.7] & \big[\ [0.1, 0.5]\ \big] \end{array},$$

$$S_4 = \begin{array}{cc} & [0.4, 0.6] \\ [0.1, 0.7] & \big[\ [0.1, 0.6]\ \big] \end{array}.$$

Then $D(S_1) = [2/5, 1]$, $D(S_2) = [1/2, 1]$, $D(S_3) = [1/5, 1]$, $D(S_4) = [1/6, 1]$, $D(A) = [7/22, 1]$.

Note that A is a complete g-IVFMFRC and $D^+(S_i) = D^+(A) = 1$ for all $i = 1, 2, 3, 4$ and $D^-(S_i) < 1$.

Theorem 7.5 *Let $A = [r_A(i)][c_A(j)][a_{ij}]_{m\times n}$ be an IVFMFRC and A^c be its complement. Then $D^+(A) + D^+(A^c) \geq 1$.*

Proof By definition $D^+(A) = \dfrac{\sum\limits_{i,j} a^+_{ij}}{\sum\limits_{i,j} r^+_A(i) \wedge c^+_A(j)}$ and $D^+(A^c) = \dfrac{\sum\limits_{i,j}(1 - a^-_{ij})}{\sum\limits_{i,j}(1 - r^-_A(i)) \wedge (1 - c^-_A(j))}$.

Now, $r^+_A(i) \wedge c^+_A(j) \leq 1$ for a fixed i and j.

Therefore, $\sum\limits_{i,j} r^+_A(i) \wedge c^+_A(j) \leq mn$.

Similarly, $\sum\limits_{i,j}(1 - r^-_A(i)) \wedge (1 - c^-_A(j)) \leq mn$.

Thus, $D^+(A) + D^+(A^c) \geq \dfrac{\sum\limits_{i,j} a^+_{ij}}{mn} + \dfrac{\sum\limits_{i,j}(1 - a^-_{ij})}{mn} = \dfrac{\sum\limits_{i,j}\{a^+_{ij} + (1 - a^-_{ij})\}}{mn} \geq 1.$

8. Application of IVFMFRC

In this section, an application of IVFMFRC is described. IVFMFRC can be successfully used to represent an image. Also, IVFMFRC is used to image contraction. Any plane image can be treated as a matrix. The grey value of a pixel is consider as an element of the corresponding matrix. The grey value depends on the colour of a pixel and it can be represented within the unit interval $[0, 1]$. Thus, the value of each element of the matrix lies on $[0, 1]$. Sometimes, it is difficult to measure the grey value of a pixel due to hesiness of the image or defect of the instrument or improper snap, etc. In this case, the grey value can be represented as a sub-interval of $[0, 1]$. Again, certain portion of the entire image may not be significant, i.e., that portion contains only background colour. While the other portion may be highly significant and some another portion is less significant, etc. Depending on this analogy, one can grade each row and each column of the matrix corresponding to the given image. Thus, the rows and columns are also uncertain and hence they have some membership values. This type of image can be represented as an IVFMFRC. Table 1 represents the IVFMFRC for the image shown in Fig.1. In this table, the first row and first column represent the membership values of the IVFMFRC.

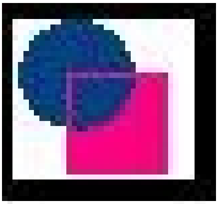

Fig. 1 A sample image

Table 1: IVFMFRC for the image of Figure 1.

Columns 1 through 4

	[0.99,0.99]	[0.99,0.99]	[0.99,0.99]	[0.98,0.99]
[0.99,0.99]	[0.04,0.05]	[0.02,0.03]	[0.03,0.04]	[0.00,0.01]
[0.28,0.30]	[0.02,0.03]	[0.01,0.02]	[0.00,0.00]	[0.00,0.01]
[0.25,0.27]	[0.05,0.06]	[0.00,0.01]	[0.02,0.03]	[0.06,0.07]
[0.24,0.25]	[0.01,0.02]	[0.00,0.01]	[0.94,0.95]	[1.00,1.00]
[0.20,0.22]	[0.03,0.04]	[0.13,0.14]	[0.98,0.99]	[0.98,0.99]
[0.11,0.12]	[0.09,0.10]	[0.03,0.04]	[1.00,1.00]	[0.98,0.99]
[0.10,0.11]	[0.01,0.02]	[0.03,0.04]	[0.91,0.92]	[0.96,0.97]
[0.08,0.10]	[0.00,0.01]	[0.00,0.01]	[0.98,0.99]	[0.95,0.96]
[0.07,0.08]	[0.01,0.02]	[0.00,0.00]	[0.99,1.00]	[0.93,0.94]
[0.05,0.06]	[0.01,0.02]	[0.00,0.01]	[0.95,0.96]	[0.95,0.96]
[0.04,0.05]	[0.01,0.02]	[0.00,0.01]	[0.97,0.98]	[0.91,0.92]
[0.02,0.04]	[0.01,0.02]	[0.01,0.02]	[0.92,0.93]	[0.12,0.13]
[0.02,0.05]	[0.00,0.01]	[0.00,0.00]	[0.87,0.87]	[0.07,0.08]
[0.02,0.04]	[0.03,0.04]	[0.00,0.00]	[0.83,0.84]	[0.11,0.12]
[0.02,0.04]	[0.00,0.01]	[0.03,0.04]	[0.89,0.90]	[0.04,0.05]
[0.02,0.03]	[0.01,0.02]	[0.00,0.00]	[0.90,0.91]	[0.14,0.15]
[0.00,0.02]	[0.03,0.04]	[0.02,0.03]	[0.96,0.97]	[0.90,0.91]
[0.00,0.00]	[0.00,0.01]	[0.00,0.01]	[0.98,0.99]	[0.94,0.95]
[0.00,0.00]	[0.01,0.02]	[0.02,0.03]	[0.98,0.99]	[0.91,0.92]
[0.00,0.00]	[0.00,0.01]	[0.00,0.00]	[0.92,0.93]	[0.96,0.97]
[0.01,0.01]	[0.01,0.02]	[0.00,0.00]	[0.98,0.99]	[0.94,0.95]
[0.01,0.02]	[0.00,0.00]	[0.00,0.00]	[0.93,0.94]	[0.99,1.00]
[0.01,0.02]	[0.00,0.01]	[0.00,0.01]	[1.00,1.00]	[0.98,0.99]
[0.03,0.04]	[0.00,0.00]	[0.02,0.03]	[0.98,0.99]	[0.99,1.00]
[0.02,0.05]	[0.00,0.00]	[0.00,0.00]	[0.98,0.99]	[0.97,0.98]
[0.15,0.20]	[0.00,0.01]	[0.00,0.00]	[0.98,0.99]	[0.96,0.97]
[0.25,0.30]	[0.00,0.01]	[0.00,0.00]	[0.99,1.00]	[0.99,1.00]
[0.32,0.40]	[0.01,0.02]	[0.00,0.01]	[0.99,1.00]	[0.99,1.00]
[0.40,0.43]	[0.00,0.01]	[0.00,0.01]	[0.99,1.00]	[0.99,1.00]
[0.44,0.48]	[0.00,0.01]	[0.00,0.00]	[1.00,1.00]	[1.00,1.00]
[0.52,0.54]	[0.00,0.01]	[0.00,0.00]	[0.99,1.00]	[0.99,1.00]
[0.66,0.70]	[0.00,0.00]	[0.00,0.00]	[0.99,1.00]	[0.99,1.00]
[0.78,0.82]	[0.00,0.00]	[0.02,0.03]	[0.98,0.99]	[1.00,1.00]
[0.84,0.85]	[0.01,0.02]	[0.00,0.00]	[0.09,0.10]	[0.00,0.00]
[0.88,0.90]	[0.02,0.03]	[0.07,0.08]	[0.00,0.00]	[0.00,0.00]
[0.85,0.88]	[0.00,0.00]	[0.03,0.04]	[0.00,0.00]	[0.01,0.02]

```
[0.86,0.87]|[0.07,0.08] [0.02,0.03] [0.00,0.01] [0.00,0.00]
[0.99,0.99]|[1.00,1.00] [0.91,0.92] [1.00,1.00] [0.99,1.00]

  Columns 5 through 8
            [0.98,0.99] [0.97,0.98] [0.97,0.98] [0.58,0.60]
------------------------------------------------------------
[0.99,0.99]|[0.03,0.04] [0.02,0.03] [0.03,0.04] [0.00,0.01]
[0.28,0.30]|[0.04,0.05] [0.02,0.03] [0.00,0.01] [0.00,0.01]
[0.25,0.27]|[0.02,0.03] [0.02,0.03] [0.00,0.00] [0.00,0.01]
[0.24,0.25]|[0.99,1.00] [0.98,0.99] [0.92,0.93] [0.94,0.95]
[0.20,0.22]|[0.97,0.98] [0.96,0.97] [0.90,0.91] [0.85,0.86]
[0.11,0.12]|[0.96,0.97] [0.83,0.84] [0.83,0.84] [0.08,0.09]
[0.10,0.11]|[0.89,0.90] [0.83,0.84] [0.05,0.06] [0.01,0.02]
[0.08,0.10]|[0.88,0.89] [0.04,0.05] [0.05,0.06] [0.00,0.01]
[0.07,0.08]|[0.89,0.90] [0.08,0.09] [0.00,0.01] [0.01,0.02]
[0.05,0.06]|[0.06,0.07] [0.05,0.06] [0.07,0.08] [0.04,0.05]
[0.04,0.05]|[0.15,0.16] [0.12,0.13] [0.00,0.00] [0.01,0.02]
[0.02,0.04]|[0.06,0.07] [0.01,0.02] [0.00,0.01] [0.01,0.02]
[0.02,0.05]|[0.04,0.05] [0.03,0.04] [0.02,0.03] [0.02,0.03]
[0.02,0.04]|[0.01,0.02] [0.00,0.01] [0.00,0.00] [0.06,0.07]
[0.02,0.04]|[0.08,0.09] [0.12,0.13] [0.04,0.05] [0.00,0.01]
[0.02,0.03]|[0.04,0.05] [0.04,0.05] [0.04,0.05] [0.06,0.07]
[0.00,0.02]|[0.10,0.11] [0.08,0.09] [0.00,0.00] [0.06,0.07]
[0.00,0.00]|[0.10,0.11] [0.09,0.10] [0.01,0.02] [0.01,0.02]
[0.00,0.00]|[0.89,0.90] [0.00,0.01] [0.12,0.13] [0.01,0.02]
[0.00,0.00]|[0.90,0.91] [0.12,0.13] [0.01,0.02] [0.01,0.02]
[0.01,0.01]|[0.93,0.94] [0.90,0.91] [0.01,0.02] [0.07,0.08]
[0.01,0.02]|[0.98,0.99] [0.95,0.96] [0.92,0.93] [0.13,0.14]
[0.01,0.02]|[0.94,0.95] [0.97,0.98] [0.98,0.99] [0.93,0.94]
[0.03,0.04]|[0.99,1.00] [0.99,1.00] [0.98,0.99] [0.98,0.99]
[0.02,0.05]|[0.95,0.96] [0.95,0.96] [0.95,0.96] [0.94,0.95]
[0.15,0.20]|[0.94,0.95] [0.93,0.94] [0.94,0.95] [0.94,0.95]
[0.25,0.30]|[0.97,0.98] [0.99,1.00] [0.98,0.99] [0.99,1.00]
[0.32,0.40]|[0.98,0.99] [1.00,1.00] [0.98,0.99] [0.99,1.00]
[0.40,0.43]|[0.98,0.99] [0.99,1.00] [0.98,0.99] [1.00,1.00]
[0.44,0.48]|[0.98,0.99] [0.99,1.00] [0.99,1.00] [0.99,1.00]
[0.52,0.54]|[0.98,0.99] [1.00,1.00] [0.99,1.00] [0.99,1.00]
[0.66,0.70]|[0.98,0.99] [0.99,1.00] [0.98,0.99] [1.00,1.00]
[0.78,0.82]|[1.00,1.00] [0.98,0.99] [0.99,1.00] [0.98,0.99]
[0.84,0.85]|[0.00,0.00] [0.00,0.00] [0.03,0.04] [0.01,0.02]
[0.88,0.90]|[0.03,0.04] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.85,0.88]|[0.02,0.03] [0.00,0.00] [0.00,0.00] [0.03,0.04]
[0.86,0.87]|[0.00,0.00] [0.00,0.00] [0.01,0.02] [0.00,0.01]
[0.99,0.99]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [0.98,0.99]

  Columns 9 through 12
            [0.58,0.60] [0.55,0.57] [0.55,0.56] [0.55,0.56]
------------------------------------------------------------
[0.99,0.99]|[0.00,0.01] [0.00,0.00] [0.03,0.04] [0.01,0.02]
[0.28,0.30]|[0.00,0.01] [0.02,0.03] [0.01,0.02] [0.01,0.02]
[0.25,0.27]|[0.00,0.00] [0.00,0.00] [0.01,0.02] [0.00,0.00]
[0.24,0.25]|[0.96,0.97] [0.95,0.96] [0.91,0.92] [0.14,0.15]
[0.20,0.22]|[0.89,0.90] [0.04,0.05] [0.15,0.16] [0.05,0.06]
```

```
[0.11,0.12]|[0.15,0.16] [0.05,0.06] [0.07,0.08] [0.09,0.10]
[0.10,0.11]|[0.00,0.00] [0.07,0.07] [0.00,0.01] [0.06,0.07]
[0.08,0.10]|[0.02,0.03] [0.03,0.04] [0.06,0.07] [0.00,0.00]
[0.07,0.08]|[0.02,0.03] [0.00,0.01] [0.00,0.01] [0.01,0.02]
[0.05,0.06]|[0.04,0.05] [0.02,0.03] [0.01,0.02] [0.02,0.03]
[0.04,0.05]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.02,0.04]|[0.04,0.05] [0.02,0.03] [0.03,0.04] [0.03,0.04]
[0.02,0.05]|[0.01,0.02] [0.00,0.01] [0.01,0.02] [0.03,0.04]
[0.02,0.04]|[0.01,0.02] [0.02,0.03] [0.07,0.08] [0.13,0.14]
[0.02,0.04]|[0.01,0.02] [0.01,0.02] [0.05,0.06] [0.07,0.08]
[0.02,0.03]|[0.01,0.02] [0.02,0.03] [0.07,0.08] [0.10,0.11]
[0.00,0.02]|[0.01,0.02] [0.03,0.04] [0.04,0.05] [0.11,0.12]
[0.00,0.00]|[0.00,0.00] [0.02,0.03] [0.02,0.03] [0.22,0.23]
[0.00,0.00]|[0.03,0.04] [0.03,0.04] [0.04,0.05] [0.06,0.07]
[0.00,0.00]|[0.00,0.01] [0.00,0.00] [0.06,0.07] [0.14,0.15]
[0.01,0.01]|[0.04,0.05] [0.05,0.06] [0.05,0.06] [0.14,0.15]
[0.01,0.02]|[0.10,0.11] [0.00,0.00] [0.16,0.17] [0.15,0.16]
[0.01,0.02]|[0.94,0.95] [0.21,0.22] [0.07,0.08] [0.18,0.19]
[0.03,0.04]|[0.97,0.98] [0.94,0.95] [0.94,0.95] [1.00,1.00]
[0.02,0.05]|[0.93,0.94] [0.94,0.95] [0.97,0.98] [1.00,1.00]
[0.15,0.20]|[0.93,0.94] [0.96,0.97] [1.00,1.00] [1.00,1.00]
[0.25,0.30]|[0.98,0.99] [0.99,1.00] [1.00,1.00] [1.00,1.00]
[0.32,0.40]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.40,0.43]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.44,0.48]|[0.99,1.00] [0.98,0.99] [1.00,1.00] [1.00,1.00]
[0.52,0.54]|[0.98,0.99] [0.97,0.98] [0.98,0.99] [1.00,1.00]
[0.66,0.70]|[0.98,0.99] [0.96,0.97] [0.98,0.99] [0.62,0.63]
[0.78,0.82]|[0.98,0.99] [0.98,0.99] [0.99,1.00] [1.00,1.00]
[0.84,0.85]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.01,0.02]
[0.88,0.90]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.01]
[0.85,0.88]|[0.00,0.01] [0.00,0.01] [0.00,0.01] [0.00,0.01]
[0.86,0.87]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.99,0.99]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]

  Columns 13 through 16
           [0.50,0.52] [0.48,0.49] [0.47,0.50] [0.40,0.44]
-----------------------------------------------------------
[0.99,0.99]|[0.01,0.02] [0.00,0.00] [0.00,0.00] [0.01,0.02]
[0.28,0.30]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.25,0.27]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.24,0.25]|[0.10,0.11] [0.07,0.08] [0.01,0.02] [0.11,0.12]
[0.20,0.22]|[0.03,0.04] [0.04,0.05] [0.13,0.14] [0.02,0.03]
[0.11,0.12]|[0.02,0.03] [0.00,0.00] [0.07,0.08] [0.03,0.04]
[0.10,0.11]|[0.00,0.00] [0.00,0.00] [0.07,0.08] [0.14,0.15]
[0.08,0.10]|[0.05,0.06] [0.06,0.07] [0.03,0.04] [0.06,0.07]
[0.07,0.08]|[0.00,0.00] [0.02,0.03] [0.05,0.06] [0.05,0.06]
[0.05,0.06]|[0.01,0.02] [0.02,0.03] [0.02,0.03] [0.01,0.02]
[0.04,0.05]|[0.04,0.05] [0.05,0.06] [0.05,0.06] [0.05,0.06]
[0.02,0.04]|[0.14,0.15] [0.10,0.11] [0.09,0.10] [0.09,0.10]
[0.02,0.05]|[0.24,0.25] [0.22,0.23] [0.22,0.23] [0.23,0.24]
[0.02,0.04]|[0.43,0.44] [0.41,0.42] [0.41,0.42] [0.43,0.44]
[0.02,0.04]|[0.37,0.38] [0.27,0.28] [0.21,0.22] [0.19,0.20]
[0.02,0.03]|[0.41,0.42] [0.27,0.28] [0.16,0.17] [0.11,0.12]
```

```
[0.00,0.02]|[0.42,0.43] [0.27,0.28] [0.19,0.20] [0.06,0.07]
[0.00,0.00]|[0.44,0.45] [0.25,0.26] [0.09,0.10] [0.08,0.09]
[0.00,0.00]|[0.42,0.43] [0.29,0.30] [0.16,0.17] [0.04,0.05]
[0.00,0.00]|[0.43,0.44] [0.21,0.22] [0.07,0.08] [0.04,0.05]
[0.01,0.01]|[0.32,0.33] [0.34,0.35] [0.14,0.15] [0.04,0.05]
[0.01,0.02]|[0.38,0.39] [0.26,0.27] [0.19,0.20] [0.07,0.08]
[0.01,0.02]|[0.44,0.45] [0.24,0.25] [0.20,0.21] [0.21,0.22]
[0.03,0.04]|[0.49,0.50] [0.52,0.53] [0.29,0.30] [0.37,0.38]
[0.02,0.05]|[0.68,0.69] [0.78,0.79] [0.77,0.78] [0.79,0.80]
[0.15,0.20]|[0.77,0.78] [0.91,0.92] [0.92,0.93] [0.94,0.95]
[0.25,0.30]|[0.79,0.80] [0.94,0.95] [0.96,0.97] [0.98,0.99]
[0.32,0.40]|[0.80,0.81] [0.95,0.96] [0.99,1.00] [1.00,1.00]
[0.40,0.43]|[0.79,0.80] [0.94,0.95] [0.98,0.99] [0.98,0.99]
[0.44,0.48]|[0.77,0.78] [0.92,0.93] [0.96,0.97] [0.97,0.98]
[0.52,0.54]|[0.72,0.73] [0.88,0.89] [0.91,0.92] [0.93,0.94]
[0.66,0.70]|[0.74,0.75] [0.76,0.77] [0.78,0.79] [0.77,0.78]
[0.78,0.82]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.84,0.85]|[0.01,0.02] [0.02,0.03] [0.02,0.03] [0.01,0.02]
[0.88,0.90]|[0.00,0.01] [0.00,0.01] [0.00,0.01] [0.00,0.01]
[0.85,0.88]|[0.00,0.01] [0.00,0.01] [0.00,0.01] [0.00,0.01]
[0.86,0.87]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.99,0.99]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]

  Columns 17 through 20
            [0.35,0.40] [0.30,0.32] [0.27,0.31] [0.27,0.30]
-----------------------------------------------------------
[0.99,0.99]|[0.00,0.00] [0.03,0.04] [0.00,0.00] [0.05,0.06]
[0.28,0.30]|[0.00,0.00] [0.00,0.00] [0.03,0.04] [0.00,0.00]
[0.25,0.27]|[0.02,0.03] [0.00,0.00] [0.00,0.00] [0.05,0.06]
[0.24,0.25]|[0.03,0.04] [0.85,0.86] [0.88,0.89] [0.87,0.88]
[0.20,0.22]|[0.10,0.11] [0.09,0.10] [0.03,0.04] [0.83,0.84]
[0.11,0.12]|[0.03,0.04] [0.03,0.04] [0.03,0.04] [0.05,0.06]
[0.10,0.11]|[0.07,0.08] [0.11,0.12] [0.04,0.05] [0.00,0.00]
[0.08,0.10]|[0.10,0.11] [0.05,0.06] [0.03,0.04] [0.02,0.03]
[0.07,0.08]|[0.08,0.09] [0.03,0.04] [0.05,0.06] [0.00,0.00]
[0.05,0.06]|[0.09,0.10] [0.03,0.04] [0.05,0.06] [0.00,0.00]
[0.04,0.05]|[0.05,0.06] [0.03,0.04] [0.10,0.11] [0.06,0.07]
[0.02,0.04]|[0.12,0.13] [0.10,0.11] [0.12,0.13] [0.12,0.13]
[0.02,0.05]|[0.26,0.27] [0.27,0.28] [0.22,0.23] [0.25,0.26]
[0.02,0.04]|[0.34,0.35] [0.43,0.44] [0.39,0.40] [0.38,0.39]
[0.02,0.04]|[0.11,0.12] [0.14,0.15] [0.16,0.17] [0.11,0.12]
[0.02,0.03]|[0.09,0.10] [0.01,0.02] [0.08,0.09] [0.09,0.10]
[0.00,0.02]|[0.05,0.06] [0.06,0.07] [0.07,0.08] [0.05,0.06]
[0.00,0.00]|[0.05,0.06] [0.03,0.04] [0.04,0.05] [0.05,0.06]
[0.00,0.00]|[0.03,0.04] [0.00,0.01] [0.00,0.01] [0.04,0.05]
[0.00,0.00]|[0.03,0.04] [0.01,0.02] [0.02,0.03] [0.05,0.06]
[0.01,0.01]|[0.04,0.05] [0.05,0.06] [0.06,0.07] [0.10,0.11]
[0.01,0.02]|[0.09,0.10] [0.14,0.15] [0.18,0.19] [0.26,0.27]
[0.01,0.02]|[0.30,0.31] [0.37,0.38] [0.44,0.45] [0.56,0.57]
[0.03,0.04]|[0.61,0.62] [0.65,0.66] [0.72,0.73] [0.83,0.84]
[0.02,0.05]|[0.83,0.84] [0.86,0.87] [0.90,0.91] [0.92,0.93]
[0.15,0.20]|[0.95,0.96] [0.96,0.97] [0.97,0.98] [0.98,0.99]
[0.25,0.30]|[0.98,0.99] [0.98,0.99] [0.98,0.99] [0.98,0.99]
```

```
[0.32,0.40]|[0.98,0.99] [0.98,0.99] [0.98,0.99] [0.97,0.98]
[0.40,0.43]|[0.97,0.98] [0.97,0.98] [0.97,0.98] [0.97,0.98]
[0.44,0.48]|[0.95,0.96] [0.94,0.95] [0.96,0.97] [0.96,0.97]
[0.52,0.54]|[0.92,0.93] [0.92,0.93] [0.94,0.95] [0.94,0.95]
[0.66,0.70]|[0.78,0.79] [0.79,0.80] [0.79,0.80] [0.79,0.80]
[0.78,0.82]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.84,0.85]|[0.01,0.02] [0.02,0.03] [0.02,0.03] [0.02,0.03]
[0.88,0.90]|[0.00,0.01] [0.00,0.01] [0.00,0.01] [0.00,0.01]
[0.85,0.88]|[0.00,0.01] [0.00,0.01] [0.00,0.01] [0.00,0.01]
[0.86,0.87]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.99,0.99]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]

  Columns 21 through 24
            [0.25,0.27] [0.20,0.22] [0.12,0.16] [0.10,0.12]
-----------------------------------------------------------
[0.99,0.99]|[0.02,0.03] [0.06,0.07] [0.04,0.05] [0.04,0.05]
[0.28,0.30]|[0.02,0.03] [0.03,0.04] [0.04,0.05] [0.04,0.05]
[0.25,0.27]|[0.00,0.00] [0.02,0.03] [0.05,0.06] [0.02,0.03]
[0.24,0.25]|[0.94,0.95] [0.97,0.98] [0.98,0.99] [0.97,0.98]
[0.20,0.22]|[0.88,0.89] [0.93,0.94] [0.96,0.97] [0.99,1.00]
[0.11,0.12]|[0.07,0.08] [0.85,0.86] [0.92,0.93] [0.98,0.99]
[0.10,0.11]|[0.08,0.09] [0.02,0.03] [0.89,0.90] [0.95,0.96]
[0.08,0.10]|[0.00,0.00] [0.03,0.04] [0.11,0.12] [0.89,0.90]
[0.07,0.08]|[0.03,0.04] [0.01,0.02] [0.12,0.13] [0.92,0.93]
[0.05,0.06]|[0.01,0.02] [0.03,0.04] [0.00,0.00] [0.20,0.21]
[0.04,0.05]|[0.03,0.04] [0.10,0.11] [0.10,0.11] [0.07,0.08]
[0.02,0.04]|[0.05,0.06] [0.06,0.07] [0.16,0.17] [0.17,0.18]
[0.02,0.05]|[0.25,0.26] [0.20,0.21] [0.21,0.22] [0.22,0.23]
[0.02,0.04]|[0.43,0.44] [0.43,0.44] [0.41,0.42] [0.44,0.45]
[0.02,0.04]|[0.09,0.10] [0.12,0.13] [0.14,0.15] [0.24,0.25]
[0.02,0.03]|[0.05,0.06] [0.07,0.08] [0.10,0.11] [0.22,0.23]
[0.00,0.02]|[0.03,0.04] [0.07,0.08] [0.12,0.13] [0.24,0.25]
[0.00,0.00]|[0.03,0.04] [0.05,0.06] [0.17,0.18] [0.37,0.38]
[0.00,0.00]|[0.05,0.06] [0.10,0.11] [0.32.0.33] [0.59,0.60]
[0.00,0.00]|[0.10,0.11] [0.23,0.24] [0.49,0.50] [0.73,0.74]
[0.01,0.01]|[0.22,0.23] [0.42,0.43] [0.67,0.68] [0.81,0.82]
[0.01,0.02]|[0.43,0.44] [0.64,0.65] [0.84,0.85] [0.91,0.92]
[0.01,0.02]|[0.70,0.71] [0.83,0.84] [0.94,0.95] [0.96,0.97]
[0.03,0.04]|[0.90,0.91] [0.90,0.91] [0.96,0.97] [0.97,0.98]
[0.02,0.05]|[0.95,0.96] [0.97,0.98] [0.99,1.00] [0.98,0.99]
[0.15,0.20]|[0.99,1.00] [0.99,1.00] [1.00,1.00] [0.99,1.00]
[0.25,0.30]|[0.98,0.99] [0.98,0.99] [0.98,0.99] [0.99,1.00]
[0.32,0.40]|[0.97,0.98] [0.97,0.98] [0.98,0.99] [0.99,1.00]
[0.40,0.43]|[0.96,0.97] [0.97,0.98] [0.98,0.99] [1.00,1.00]
[0.44,0.48]|[0.94,0.95] [0.95,0.96] [0.97,0.98] [0.97,0.98]
[0.52,0.54]|[0.94,0.95] [1.00,1.00] [0.94,0.95] [0.93,0.94]
[0.66,0.70]|[0.79,0.80] [1.00,1.00] [0.79,0.80] [0.78,0.79]
[0.78,0.82]|[1.00,1.00] [0.00,0.01] [1.00,1.00] [1.00,1.00]
[0.84,0.85]|[0.02,0.03] [0.00,0.00] [0.02,0.03] [0.02,0.03]
[0.88,0.90]|[0.00,0.01] [0.00,0.01] [0.00,0.01] [0.00,0.01]
[0.85,0.88]|[0.00,0.01] [0.00,0.00] [0.00,0.01] [0.00,0.01]
[0.86,0.87]|[0.00,0.00] [1.00,1.00] [0.00,0.00] [0.00,0.00]
[0.99,0.99]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
```

```
  Columns 25 through 28
            [0.10,0.12] [0.09,0.10] [0.08,0.10] [0.07,0.09]
-----------------------------------------------------------
[0.99,0.99]|[0.04,0.05] [0.04,0.05] [0.02,0.03] [0.01,0.02]
[0.28,0.30]|[0.03,0.04] [0.03,0.04] [0.01,0.02] [0.01,0.02]
[0.25,0.27]|[0.00,0.01] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.24,0.25]|[0.99,1.00] [0.96,0.97] [0.94,0.95] [0.92,0.93]
[0.20,0.22]|[0.99,1.00] [0.97,0.98] [0.95,0.96] [0.94,0.95]
[0.11,0.12]|[0.99,1.00] [0.99,1.00] [0.97,0.98] [0.96,0.99]
[0.10,0.11]|[0.98,0.99] [0.99,1.00] [0.99,1.00] [0.97,0.98]
[0.08,0.10]|[0.97,0.98] [0.99,1.00] [0.99,1.00] [0.99,1.00]
[0.07,0.08]|[0.97,0.98] [0.99,1.00] [0.96,0.97] [1.00,1.00]
[0.05,0.06]|[0.96,0.97] [0.99,1.00] [1.00,1.00] [0.99,1.00]
[0.04,0.05]|[0.94,0.95] [0.99,1.00] [1.00,1.00] [1.00,1.00]
[0.02,0.04]|[0.20,0.21] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.02,0.05]|[0.33,0.34] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.02,0.04]|[0.49,0.50] [0.61,0.62] [0.61,0.62] [0.69,0.70]
[0.02,0.04]|[0.38,0.39] [0.65,0.66] [0.83,0.84] [0.85,0.86]
[0.02,0.03]|[0.40,0.41] [0.76,0.77] [0.87,0.88] [0.95,0.96]
[0.00,0.02]|[0.66,0.67] [0.82,0.83] [0.93,0.94] [0.97,0.98]
[0.00,0.00]|[0.73,0.74] [0.87,0.88] [0.96,0.97] [0.98,0.99]
[0.00,0.00]|[0.80,0.81] [0.90,0.91] [0.97,0.98] [0.98,0.99]
[0.00,0.00]|[0.85,0.86] [0.91,0.92] [0.97,0.98] [0.98,0.99]
[0.01,0.01]|[0.89,0.90] [0.90,0.91] [0.96,0.97] [0.98,0.99]
[0.01,0.02]|[0.90,0.91] [0.91,0.92] [0.96,0.97] [0.97,0.98]
[0.01,0.02]|[0.91,0.92] [0.90,0.91] [0.95,0.96] [0.97,0.98]
[0.03,0.04]|[0.93,0.94] [0.92,0.93] [0.96,0.97] [0.98,0.99]
[0.02,0.05]|[0.97,0.98] [0.95,0.96] [0.97,0.98] [0.98,0.99]
[0.15,0.20]|[0.98,0.99] [0.96,0.97] [0.98,0.99] [0.97,0.98]
[0.25,0.30]|[0.99,1.00] [0.98,0.99] [0.96,0.97] [0.95,0.96]
[0.32,0.40]|[0.99,1.00] [0.98,0.99] [0.96,0.97] [0.95,0.96]
[0.40,0.43]|[1.00,1.00] [0.98,0.99] [0.96,0.97] [0.95,0.96]
[0.44,0.48]|[0.96,0.97] [0.95,0.96] [0.95,0.96] [0.93,0.94]
[0.52,0.54]|[0.90,0.91] [0.89,0.90] [0.90,0.91] [0.90,0.91]
[0.66,0.70]|[0.75,0.76] [0.74,0.75] [0.76,0.77] [0.76,0.77]
[0.78,0.82]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.84,0.85]|[0.01,0.02] [0.01,0.02] [0.02,0.03] [0.02,0.03]
[0.88,0.90]|[0.00,0.01] [0.00,0.01] [0.00,0.01] [0.00,0.01]
[0.85,0.88]|[0.00,0.01] [0.00,0.01] [0.00,0.01] [0.00,0.01]
[0.86,0.87]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.99,0.99]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]

  Columns 29 through 32
            [0.05,0.06] [0.04,0.06] [0.02,0.04] [0.01,0.02]
-----------------------------------------------------------
[0.99,0.99]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.28,0.30]|[0.00,0.01] [0.00,0.00] [0.00,0.01] [0.01,0.02]
[0.25,0.27]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.01,0.02]
[0.24,0.25]|[0.94,0.95] [0.97,0.98] [0.99,1.00] [1.00,1.00]
[0.20,0.22]|[0.96,0.97] [0.98,0.99] [0.98,0.99] [0.99,1.00]
[0.11,0.12]|[0.97,0.98] [0.99,1.00] [0.99,1.00] [0.99,1.00]
[0.10,0.11]|[0.97,0.98] [0.98,0.99] [0.99,1.00] [0.99,1.00]
```

```
[0.08,0.10]|[0.98,0.99] [0.97,0.98] [0.98,0.99] [0.98,0.99]
[0.07,0.08]|[1.00,1.00] [0.98,0.99] [1.00,1.00] [0.97,0.98]
[0.05,0.06]|[0.98,0.99] [1.00,1.00] [0.96,0.97] [1.00,1.00]
[0.04,0.05]|[1.00,1.00] [1.00,1.00] [0.94,0.95] [0.98,0.99]
[0.02,0.04]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [0.92,0.93]
[0.02,0.05]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.02,0.04]|[0.74,0.75] [0.64,0.65] [0.50,0.51] [1.00,1.00]
[0.02,0.04]|[0.81,0.82] [0.80,0.81] [0.65,0.66] [1.00,1.00]
[0.02,0.03]|[0.94,0.95] [0.85,0.86] [0.70,0.71] [1.00,1.00]
[0.00,0.02]|[0.96,0.97] [0.85,0.86] [0.70,0.71] [1.00,1.00]
[0.00,0.00]|[0.97,0.98] [0.85,0.86] [0.70,0.71] [1.00,1.00]
[0.00,0.00]|[0.96,0.97] [0.84,0.85] [0.69,0.70] [1.00,1.00]
[0.00,0.00]|[0.96,0.97] [0.83,0.84] [0.69,0.70] [1.00,1.00]
[0.01,0.01]|[0.96,0.97] [0.84,0.85] [0.68,0.69] [1.00,1.00]
[0.01,0.02]|[0.96,0.97] [0.84,0.85] [0.69,0.70] [1.00,1.00]
[0.01,0.02]|[0.96,0.97] [0.85,0.86] [0.69,0.70] [1.00,1.00]
[0.03,0.04]|[0.97,0.98] [0.85,0.86] [0.70,0.71] [1.00,1.00]
[0.02,0.05]|[0.97,0.98] [0.85,0.86] [0.70,0.71] [1.00,1.00]
[0.15,0.20]|[0.96,0.97] [0.85,0.86] [0.70,0.71] [1.00,1.00]
[0.25,0.30]|[0.95,0.96] [0.83,0.84] [0.69,0.70] [1.00,1.00]
[0.32,0.40]|[0.94,0.95] [0.82,0.83] [0.70,0.71] [1.00,1.00]
[0.40,0.43]|[0.93,0.94] [0.81,0.82] [0.70,0.71] [1.00,1.00]
[0.44,0.48]|[0.92,0.93] [0.81,0.82] [0.69.0.70] [1.00,1.00]
[0.52,0.54]|[0.90,0.91] [0.79,0.80] [0.68,0.69] [1.00,1.00]
[0.66,0.70]|[0.77,0.78] [0.68,0.69] [0.60,0.61] [1.00,1.00]
[0.78,0.82]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.84,0.85]|[0.01,0.02] [0.01,0.02] [0.01,0.02] [0.00,0.01]
[0.88,0.90]|[0.00,0.01] [0.00,0.01] [0.00,0.00] [0.00,0.00]
[0.85,0.88]|[0.00,0.01] [0.00,0.01] [0.00,0.01] [0.00,0.01]
[0.86,0.87]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.01]
[0.99,0.99]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]

Columns 33 through 36
            [0.00,0.01] [0.08,0.09] [0.11,0.13] [0.14,0.15]
-----------------------------------------------------------
[0.99,0.99]|[0.01,0.02] [0.00,0.01] [0.00,0.00] [0.00,0.00]
[0.28,0.30]|[0.00,0.01] [0.00,0.01] [0.07,0.08] [0.02,0.03]
[0.25,0.27]|[0.05,0.06] [0.00,0.01] [0.00,0.00] [0.05,0.06]
[0.24,0.25]|[0.99,1.00] [1.00,1.00] [1.00,1.00] [0.95,0.96]
[0.20,0.22]|[1.00,1.00] [1.00,1.00] [0.96,0.97] [1.00,1.00]
[0.11,0.12]|[1.00,1.00] [0.98,0.99] [1.00,1.00] [0.98,0.99]
[0.10,0.11]|[0.99,1.00] [0.98,0.99] [1.00,1.00] [1.00,1.00]
[0.08,0.10]|[0.98,0.99] [1.00,1.00] [0.96,0.97] [1.00,1.00]
[0.07,0.08]|[0.99,1.00] [0.99,1.00] [1.00,1.00] [1.00,1.00]
[0.05,0.06]|[1.00,1.00] [0.99,1.00] [1.00,1.00] [1.00,1.00]
[0.04,0.05]|[0.99,1.00] [0.99,1.00] [1.00,1.00] [1.00,1.00]
[0.02,0.04]|[1.00,1.00] [0.99,1.00  [1.00,1.00] [1.00,1.00]
[0.02,0.05]|[1.00,1.00] [0.99,1.00] [1.00,1.00] [1.00,1.00]
[0.02,0.04]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.02,0.04]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.02,0.03]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.00,0.02]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.00,0.00]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
```

```
[0.00,0.00]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.00,0.00]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.01,0.01]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.01,0.02]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.01,0.02]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.03,0.04]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.02,0.05]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.15,0.20]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.25,0.30]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.32,0.40]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.40,0.43]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.44,0.48]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.52,0.54]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.66,0.70]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.78,0.82]|[1.00,1.00] [1.00,1.00] [0.98,0.99] [0.95,0.96]
[0.84,0.85]|[0.02,0.03] [0.00,0.00] [0.00,0.00] [0.05,0.06]
[0.88,0.90]|[0.00,0.00] [0.01,0.02] [0.04,0.05] [0.00,0.00]
[0.85,0.88]|[0.00,0.00] [0.03,0.04] [0.00,0.00] [0.00,0.00]
[0.86,0.87]|[0.03,0.04] [0.01,0.02] [0.00,0.00] [0.02,0.03]
[0.99,0.99]|[0.97,0.98] [0.96,0.97] [1.00,1.00] [1.00,1.00]

  Columns 37 through 40
            [0.19,0.20] [0.29,0.30] [0.58,0.60] [0.60,0.62]
-----------------------------------------------------------
[0.99,0.99]|[0.02,0.03] [0.04,0.05] [0.00,0.00] [0.00,0.00]
[0.28,0.30]|[0.00,0.00] [0.00,0.00] [0.01,0.02] [0.02,0.03]
[0.25,0.27]|[0.00,0.00] [0.05,0.06] [0.07,0.08] [0.00,0.00]
[0.24,0.25]|[0.02,0.03] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.20,0.22]|[0.00,0.01] [0.06,0.07] [0.00,0.00] [0.01,0.02]
[0.11,0.12]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.10,0.11]|[0.01,0.02] [0.00,0.00] [0.03,0.04] [0.02,0.03]
[0.08,0.10]|[0.00,0.00] [0.02,0.03] [0.00,0.00] [0.00,0.01]
[0.07,0.08]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.05,0.06]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.04,0.05]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.02,0.04]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.02,0.05]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.02,0.04]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.02,0.04]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.02,0.03]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.00,0.02]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.00,0.00]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.00,0.00]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.00,0.00]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.01,0.01]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.01,0.02]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.01,0.02]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.03,0.04]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.02,0.05]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.15,0.20]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.25,0.30]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.32,0.40]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.40,0.43]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
```

[0.44,0.48]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]
[0.52,0.54]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]
[0.66,0.70]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]	[0.00,0.00]
[0.78,0.82]	[0.03,0.04]	[0.00,0.01]	[0.00,0.00]	[0.00,0.01]
[0.84,0.85]	[0.00,0.00]	[0.03,0.04]	[0.04,0.05]	[0.00,0.00]
[0.88,0.90]	[0.02,0.03]	[0.00,0.00]	[0.00,0.00]	[0.01,0.02]
[0.85,0.88]	[0.02,0.03]	[0.03,0.04]	[0.00,0.00]	[0.04,0.05]
[0.86,0.87]	[0.00,0.00]	[0.04,0.05]	[0.00,0.00]	[0.00,0.00]
[0.99,0.99]	[0.96,0.97]	[0.97,0.98]	[1.00,1.00]	[1.00,1.00]

If the membership value of a row or a column is zero, then that row and column may be removed to reduce the size of the matrix/image.

The complement of the image of Figure 1 is shown in Figure 2. This is obtained by finding the complement of the IVFMFRC shown in Table 1.

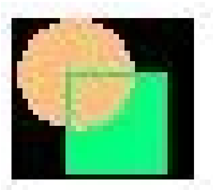

Fig. 2 Complement image

Table 2: IVFMFRC for the image of Figure 2.

Columns 1 through 4

	[0.01,0.01]	[0.01,0.01]	[0.01,0.01]	[0.01,0.02]
[0.01,0.01]	[0.95,0.96]	[0.97,0.98]	[0.96,0.96]	[0.99,0.99]
[0.70,0.72]	[0.97,0.98]	[0.98,0.99]	[1.00,1.00]	[0.99,1.00]
[0.73,0.75]	[0.94,0.95]	[0.99,1.00]	[0.97,0.97]	[0.93,0.93]
[0.75,0.76]	[0.97,0.98]	[0.99,1.00]	[0.05,0.05]	[0.00,0.00]
[0.78,0.80]	[0.96,0.96]	[0.86,0.87]	[0.01,0.02]	[0.01,0.02]
[0.88,0.89]	[0.90,0.91]	[0.96,0.96]	[0.00,0.00]	[0.01,0.01]
[0.89,0.90]	[0.98,0.99]	[0.96,0.96]	[0.08,0.08]	[0.03,0.03]
[0.90,0.92]	[0.99,0.99]	[0.99,1.00]	[0.01,0.02]	[0.04,0.04]
[0.92,0.93]	[0.98,0.99]	[1.00,1.00]	[0.00,0.01]	[0.06,0.06]
[0.94,0.95]	[0.98,1.00]	[0.99,1.00]	[0.04,0.05]	[0.04,0.05]
[0.95,0.96]	[0.98,0.99]	[0.99,0.99]	[0.02,0.03]	[0.08,0.09]
[0.96,0.98]	[0.98,0.99]	[0.98,0.99]	[0.07,0.07]	[0.87,0.87]
[0.95,0.98]	[0.99,1.00]	[1.00,1.00]	[0.12,0.13]	[0.92,0.93]
[0.96,0.98]	[0.96,0.96]	[1.00,1.00]	[0.16,0.16]	[0.88,0.89]
[0.96,0.98]	[0.99,1.00]	[0.96,0.97]	[0.10,0.12]	[0.95,0.95]
[0.97,0.98]	[0.98,0.98]	[1.00,1.00]	[0.09,0.09]	[0.85,0.85]
[0.98,1.00]	[0.96,0.97]	[0.97,0.97]	[0.03,0.04]	[0.09,0.09]
[1.00,1.00]	[0.99,0.99]	[0.99,1.00]	[0.01,0.02]	[0.05,0.06]
[1.00,1.00]	[0.98,0.98]	[0.97,0.97]	[0.01,0.02]	[0.08,0.09]
[1.00,1.00]	[0.99,1.00]	[1.00,1.00]	[0.07,0.08]	[0.03,0.04]

```
 [0.99,0.99]|[0.98,0.99] [1.00,1.00] [0.01,0.01] [0.05,0.05]
 [0.98,0.99]|[1.00,1.00] [1.00,1.00] [0.06,0.06] [0.00,0.01]
 [0.98,0.99]|[0.99,0.99] [0.99,0.99] [0.00,0.00] [0.01,0.02]
 [0.96,0.97]|[1.00,1.00] [0.97,0.98] [0.01,0.02] [0.00,0.00]
 [0.95,0.98]|[1.00,1.00] [1.00,1.00] [0.01,0.02] [0.02,0.02]
 [0.80,0.85]|[0.99,1.00] [1.00,1.00] [0.01,0.02] [0.03,0.04]
 [0.70,0.75]|[0.99,0.99] [1.00,1.00] [0.00,0.00] [0.00,0.00]
 [0.60,0.68]|[0.98,0.99] [0.99,0.99] [0.00,0.01] [0.00,0.00]
 [0.57,0.60]|[0.99,0.99] [0.99,0.99] [0.00,0.00] [0.00,0.00]
 [0.52,0.56]|[0.99,0.99] [1.00,1.00] [0.00,0.00] [0.00,0.00]
 [0.46,0.48]|[0.99,1.00] [1.00,1.00] [0.00,0.01] [0.00,0.01]
 [0.30,0.34]|[1.00,1.00] [1.00,1.00] [0.00,0.01] [0.00,0.01]
 [0.18,0.22]|[1.00,1.00] [0.97,0.98] [0.01,0.01] [0.00,0.00]
 [0.15,0.16]|[0.98,0.98] [1.00,1.00] [0.90,0.91] [1.00,1.00]
 [0.10,0.12]|[0.97,0.98] [0.92,0.93] [1.00,1.00] [1.00,1.00]
 [0.15,0.18]|[1.00,1.00] [0.96,0.97] [1.00,1.00] [0.98,0.98]
 [0.13,0.14]|[0.92,0.93] [0.97,0.97] [0.99,0.99] [1.00,1.00]
 [0.01,0.01]|[0.00,0.00] [0.08,0.08] [0.00,0.00] [0.00,0.00]

  Columns 5 through 8
           [0.01,0.02] [0.02,0.03] [0.02,0.03] [0.40,0.42]
-----------------------------------------------------------
[0.01,0.01]|[0.96,0.97] [0.97,0.98] [0.960.96]  [0.99,0.99]
[0.70,0.72]|[0.95,0.96] [0.97,0.97] [0.99,0.99] [0.99,0.99]
[0.73,0.75]|[0.97,0.97] [0.97,0.97] [1.00,1.00] [0.99,0.99]
[0.75,0.76]|[0.00,0.00] [0.01,0.02] [0.07,0.07] [0.05,0.05]
[0.78,0.80]|[0.02,0.03] [0.03,0.04] [0.09,0.09] [0.14,0.14]
[0.88,0.89]|[0.03,0.04] [0.16,0.16] [0.16,0.16] [0.91,0.91]
[0.89,0.90]|[0.10,0.11] [0.16,0.16] [0.94,0.95] [0.98,0.98]
[0.90,0.92]|[0.11,0.11] [0.95,0.96] [0.94,0.95] [0.99,0.99]
[0.92,0.93]|[0.10,0.11] [0.91,0.91] [0.99,0.99] [0.98,0.98]
[0.94,0.95]|[0.93,0.93] [0.94,0.94] [0.92,0.93] [0.95,0.96]
[0.95,0.96]|[0.84,0.85] [0.87,0.88] [1.00,1.00] [0.98,0.99]
[0.96,0.98]|[0.93,0.94] [0.98,0.99] [0.99,1.00] [0.98,0.98]
[0.95,0.98]|[0.95,0.96] [0.96,0.96] [0.97,0.97] [0.97,0.98]
[0.96,0.98]|[0.98,0.98] [0.99,1.00] [1.00,1.00] [0.93,0.93]
[0.96,0.98]|[0.91,0.92] [0.87,0.88] [0.95,0.95] [0.99,1.00]
[0.97,0.98]|[0.95,0.95] [0.95,0.95] [0.95,0.95] [0.93,0.93]
[0.98,1.00]|[0.89,0.90] [0.91,0.92] [1.00,1.00] [0.93,0.94]
[1.00,1.00]|[0.89,0.89] [0.90,0.91] [0.98,0.98] [0.98,0.98]
[1.00,1.00]|[0.10,0.11] [0.99,1.00] [0.87,0.87] [0.98,0.98]
[1.00,1.00]|[0.09,0.09] [0.87,0.87] [0.98,0.98] [0.98,0.98]
[0.99,0.99]|[0.06,0.06] [0.09,0.09] [0.98,0.98] [0.92,0.92]
[0.98,0.99]|[0.01,0.02] [0.04,0.43] [0.07,0.07] [0.86,0.86]
[0.98,0.99]|[0.05,0.05] [0.02,0.02] [0.01,0.02] [0.06,0.06]
[0.96,0.97]|[0.00,0.00] [0.00,0.00] [0.01,0.02] [0.01,0.02]
[0.95,0.98]|[0.04,0.05] [0.04,0.04] [0.04,0.05] [0.05,0.05]
[0.80,0.85]|[0.05,0.06] [0.06,0.06] [0.05,0.06] [0.05,0.05]
[0.70,0.75]|[0.02,0.03] [0.00,0.00] [0.01,0.01] [0.00,0.01]
[0.60,0.68]|[0.01,0.02] [0.00,0.00] [0.01,0.01] [0.00,0.00]
[0.57,0.60]|[0.01,0.02] [0.00,0.00] [0.01,0.01] [0.00,0.00]
[0.52,0.56]|[0.01,0.02] [0.00,0.00] [0.00,0.01] [0.00,0.00]
[0.46,0.48]|[0.01,0.02] [0.00,0.00] [0.00,0.01] [0.00,0.01]
```

```
[0.30,0.34]|[0.01,0.01] [0.00,0.00] [0.01,0.01] [0.00,0.00]
[0.18,0.22]|[0.00,0.00] [0.01,0.02] [0.00,0.00] [0.01,0.02]
[0.15,0.16]|[1.00,1.00] [1.00,1.00] [0.96,0.97] [0.98,0.99]
[0.10,0.12]|[0.96,0.96] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.15,0.18]|[0.97,0.98] [1.00,1.00] [1.00,1.00] [0.96,0.97]
[0.13,0.14]|[1.00,1.00] [1.00,1.00] [0.98,0.99] [0.99,0.99]
[0.01,0.01]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.01,0.02]

 Columns 9 through 12
            [0.40,0.42] [0.43,0.45] [0.44,0.45] [0.44,0.45]
------------------------------------------------ -----------
[0.01,0.01]|[0.99,0.99] [1.00,1.00] [0.96,0.96] [0.98,0.99]
[0.70,0.72]|[0.99,1.00] [0.97,0.97] [0.98,0.99] [0.98,0.98]
[0.73,0.75]|[1.00,1.00] [1.00,1.00] [0.98,0.98] [1.00,1.00]
[0.75,0.76]|[0.03,0.04] [0.04,0.05] [0.08,0.08] [0.85,0.85]
[0.78,0.80]|[0.10,0.12] [0.95,0.96] [0.84,0.84] [0.94,0.94]
[0.88,0.89]|[0.84,0.85] [0.94,0.94] [0.92,0.93] [0.90,0.91]
[0.89,0.90]|[1.00,1.00] [0.92,0.93] [0.99,0.99] [0.93,0.94]
[0.90,0.92]|[0.97,0.97] [0.96,0.96] [0.93,0.94] [1.00,1.00]
[0.92,0.93]|[0.97,0.97] [0.99,1.00] [0.99,1.00] [0.98,0.98]
[0.94,0.95]|[0.95,0.96] [0.97,0.98] [0.98,0.99] [0.97,0.98]
[0.95,0.96]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [0.99,0.99]
[0.96,0.98]|[0.95,0.96] [0.97,0.97] [0.96,0.96] [0.96,0.96]
[0.95,0.98]|[0.98,0.98] [0.99,1.00] [0.98,0.98] [0.96,0.96]
[0.96,0.98]|[0.98,0.98] [0.97,0.97] [0.92,0.93] [0.86,0.87]
[0.96,0.98]|[0.98,0.99] [0.98,0.98] [0.94,0.95] [0.92,0.92]
[0.97,0.98]|[0.98,0.98] [0.97,0.97] [0.92,0.92] [0.89,0.89]
[0.98,1.00]|[0.98,0.98] [0.96,0.96] [0.95,0.96] [0.88,0.88]
[1.00,1.00]|[1.00,1.00] [0.97,0.98] [0.97,0.97] [0.77,0.78]
[1.00,1.00]|[0.96,0.97] [0.96,0.96] [0.95,0.96] [0.93,0.93]
[1.00,1.00]|[0.99,1.00] [1.00,1.00] [0.93,0.93] [0.85,0.86]
[0.99,0.99]|[0.95,0.96] [0.94,0.95] [0.94,0.95] [0.85,0.86]
[0.98,0.99]|[0.89,0.89] [1.00,1.00] [0.83,0.84] [0.84,0.84]
[0.98,0.99]|[0.05,0.05] [0.78,0.79] [0.92,0.92] [0.81,0.82]
[0.96,0.97]|[0.02,0.03] [0.05,0.05] [0.05,0.05] [0.00,0.00]
[0.95,0.98]|[0.06,0.06] [0.05,0.05] [0.02,0.02] [0.00,0.00]
[0.80,0.85]|[0.06,0.07] [0.03,0.04] [0.00,0.00] [0.00,0.00]
[0.70,0.75]|[0.01,0.02] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.60,0.68]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.57,0.60]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.52,0.56]|[0.00,0.01] [0.01,0.01] [0.00,0.00] [0.00,0.00]
[0.46,0.40]|[0.01,0.02] [0.02,0.03] [0.01,0.02] [0.00,0.00]
[0.30,0.34]|[0.01,0.02] [0.03,0.03] [0.01,0.02] [0.00,0.00]
[0.18,0.22]|[0.01,0.01] [0.01,0.02] [0.00,0.00] [0.00,0.00]
[0.15,0.16]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [0.99,1.00]
[0.10,0.12]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.15,0.18]|[0.99,0.99] [0.99,0.99] [0.99,0.99] [0.99,0.99]
[0.13,0.14]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.01,0.01]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]

  Columns 13 through 16
            [0.48,0.50] [0.49,0.52] [0.50,0.53] [0.56,0.60]
------------------------------------------------------------
```

```
[0.01,0.01]|[0.98,0.99] [1.00,1.00] [1.00,1.00] [0.98,0.99]
[0.70,0.72]|[1.00,1.00] [1.00,1.00] [0.99,0.99] [1.00,1.00]
[0.73,0.75]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.75,0.76]|[0.89,0.89] [0.92,0.92] [0.98,0.98] [0.88,0.87]
[0.78,0.80]|[0.96,0.96] [0.95,0.96] [0.86,0.86] [0.97,0.97]
[0.88,0.89]|[0.97,0.98] [0.99,0.99] [0.92,0.93] [0.96,0.96]
[0.89,0.90]|[1.00,1.00] [1.00,1.00] [0.92,0.93] [0.85,0.85]
[0.90,0.92]|[0.94,0.95] [0.93,0.94] [0.96,0.96] [0.93,0.93]
[0.92,0.93]|[1.00,1.00] [0.97,0.97] [0.94,0.95] [0.94,0.95]
[0.94,0.95]|[0.98,0.99] [0.97,0.97] [0.97,0.98] [0.98,0.99]
[0.95,0.96]|[0.95,0.96] [0.94,0.95] [0.94,0.95] [0.94,0.95]
[0.96,0.98]|[0.85,0.86] [0.89,0.89] [0.90,0.9.] [0.90,0.90]
[0.95,0.98]|[0.75,0.75] [0.77,0.78] [0.77,0.77] [0.76,0.77]
[0.96,0.98]|[0.56,0.56] [0.58,0.58] [0.58,0.58] [0.56,0.56]
[0.96,0.98]|[0.62,0.63] [0.72,0.72] [0.78,0.79] [0.80,0.81]
[0.97,0.98]|[0.58,0.58] [0.72,0.72] [0.83,0.84] [0.88,0.88]
[0.98,1.00]|[0.57,0.57] [0.72,0.72] [0.80,0.80] [0.93,0.93]
[1.00,1.00]|[0.55,0.56] [0.74,0.75] [0.90,0.90] [0.91,0.91]
[1.00,1.00]|[0.57,0.57] [0.70,0.71] [0.83,0.83] [0.95,0.95]
[1.00,1.00]|[0.56,0.56] [0.78,0.78] [0.92,0.93] [0.95,0.95]
[0.99,0.99]|[0.67,0.67] [0.65,0.65] [0.85,0.86] [0.95,0.96]
[0.98,0.99]|[0.61,0.61] [0.73,0.73] [0.80,0.81] [0.92,0.93]
[0.98,0.99]|[0.55,0.55] [0.75,0.75] [0.80,0.80] [0.78,0.78]
[0.96,0.97]|[0.50,0.51] [0.47,0.48] [0.70,0.71] [0.62,0.63]
[0.95,0.98]|[0.31,0.31] [0.21,0.21] [0.22,0.22] [0.20,0.21]
[0.80,0.85]|[0.22,0.23] [0.08,0.08] [0.07,0.07] [0.05,0.05]
[0.70,0.75]|[0.20,0.21] [0.05,0.06] [0.03,0.04] [0.01,0.02]
[0.60,0.68]|[0.20,0.20] [0.04,0.04] [0.00,0.01] [0.00,0.00]
[0.57,0.60]|[0.20,0.21] [0.05,0.06] [0.01,0.02] [0.01,0.01]
[0.52,0.56]|[0.22,0.23] [0.07,0.07] [0.03,0.04] [0.02,0.03]
[0.46,0.40]|[0.27,0.27] [0.11,0.12] [0.08,0.08] [0.06,0.07]
[0.30,0.34]|[0.37,0.37] [0.25,0.26] [0.23,0.23] [0.21,0.22]
[0.18,0.22]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.15,0.16]|[0.98,0.99] [0.98,0.98] [0.97,0.98] [0.97,0.98]
[0.10,0.12]|[0.99,1.00] [0.99,1.00] [0.99,1.00] [0.99,1.00]
[0.15,0.18]|[0.99,0.99] [0.99,0.99] [0.99,0.99] [0.99,0.99]
[0.13,0.14]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.01,0.01]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]

 Columns 17 through 20
            [0.60,0.65] [0.68,0.70] [0.69,0.73] [0.70,0.73]
-----------------------------------------------------------
[0.01,0.01]|[0.99,0.99] [0.96,0.96] [0.99,0.99] [0.94,0.95]
[0.70,0.72]|[1.00,1.00] [1.00,1.00] [0.96,0.96] [0.99,1.00]
[0.73,0.75]|[0.97,0.98] [1.00,1.00] [1.00,1.00] [0.94,0.95]
[0.75,0.76]|[0.96,0.96] [0.14,0.15] [0.11,0.12] [0.12,0.13]
[0.78,0.80]|[0.89,0.89] [0.90,0.91] [0.96,0.96] [0.16,0.16]
[0.88,0.89]|[0.96,0.96] [0.96,0.96] [0.96,0.96] [0.94,0.95]
[0.89,0.90]|[0.92,0.93] [0.88,0.88] [0.95,0.95] [0.99,0.99]
[0.90,0.92]|[0.89,0.89] [0.94,0.94] [0.96,0.96] [0.97,0.98]
[0.92,0.93]|[0.91,0.92] [0.96,0.97] [0.94,0.95] [1.00,1.00]
[0.94,0.95]|[0.90,0.90] [0.96,0.96] [0.94,0.94] [1.00,1.00]
[0.95,0.96]|[0.94,0.94] [0.96,0.96] [0.89,0.90] [0.93,0.94]
```

```
[0.96,0.98]|[0.87,0.88] [0.89,0.90] [0.87,0.87] [0.87,0.87]
[0.95,0.98]|[0.73,0.74] [0.72,0.73] [0.77,0.78] [0.74,0.75]
[0.96,0.98]|[0.65,0.65] [0.56,0.57] [0.60,0.61] [0.61,0.62]
[0.96,0.98]|[0.88,0.89] [0.85,0.85] [0.83,0.83] [0.88,0.89]
[0.97,0.98]|[0.90,0.90] [0.98,0.98] [0.91,0.92] [0.90,0.90]
[0.98,1.00]|[0.94,0.94] [0.93,0.94] [0.92,0.93] [0.94,0.95]
[1.00,1.00]|[0.94,0.95] [0.96,0.97] [0.95,0.96] [0.94,0.95]
[1.00,1.00]|[0.96,0.97] [0.99,1.00] [0.99,0.99] [0.95,0.96]
[1.00,1.00]|[0.96,0.96] [0.98,0.98] [0.97,0.97] [0.94,0.94]
[0.99,0.99]|[0.95,0.96] [0.94,0.94] [0.93,0.93] [0.89,0.89]
[0.98,0.99]|[0.90,0.90] [0.85,0.86] [0.81,0.81] [0.73,0.73]
[0.98,0.99]|[0.69,0.70] [0.62,0.63] [0.55,0.55] [0.43,0.44]
[0.96,0.97]|[0.38,0.39] [0.34,0.35] [0.27,0.27] [0.16,0.16]
[0.95,0.98]|[0.16,0.16] [0.13,0.13] [0.09,0.10] [0.07,0.07]
[0.80,0.85]|[0.04,0.04] [0.03,0.03] [0.02,0.02] [0.01,0.02]
[0.70,0.75]|[0.01,0.02] [0.01,0.02] [0.01,0.02] [0.01,0.02]
[0.60,0.68]|[0.01,0.01] [0.01,0.02] [0.01,0.02] [0.02,0.02]
[0.57,0.60]|[0.02,0.02] [0.02,0.03] [0.02,0.03] [0.02,0.03]
[0.52,0.56]|[0.04,0.04] [0.05,0.05] [0.03,0.04] [0.03,0.04]
[0.46,0.40]|[0.07,0.08] [0.07,0.07] [0.05,0.06] [0.05,0.06]
[0.30,0.34]|[0.22,0.22] [0.21,0.22] [0.20,0.21] [0.20,0.20]
[0.18,0.22]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.15,0.16]|[0.98,0.98] [0.98,0.98] [0.97,0.98] [0.97,0.98]
[0.10,0.12]|[0.99,1.00] [0.99,1.00] [0.99,1.00] [0.99,1.00]
[0.15,0.18]|[0.99,0.99] [0.99,0.99] [0.99,0.99] [0.99,0.99]
[0.13,0.14]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.01,0.01]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]

  Columns 21 through 24
            [0.73,0.75] [0.78,0.80] [0.84,0.88] [0.88,0.90]
------------------------------------------------------------
[0.01,0.01]|[0.97,0.98] [0.93,0.93] [0.95,0.96] [0.95,0.95]
[0.70,0.72]|[0.97,0.98] [0.96,0.96] [0.95,0.96] [0.95,0.95]
[0.73,0.75]|[0.99,0.99] [0.97,0.98] [0.94,0.95] [0.97,0.97]
[0.75,0.76]|[0.05,0.05] [0.02,0.02] [0.01,0.01] [0.02,0.03]
[0.78,0.80]|[0.11,0.11] [0.06,0.07] [0.03,0.04] [0.00,0.00]
[0.88,0.89]|[0.92,0.93] [0.14,0.14] [0.07,0.08] [0.01,0.02]
[0.89,0.90]|[0.91,0.92] [0.97,0.97] [0.10,0.11] [0.04,0.04]
[0.90,0.92]|[1.00,1.00] [0.96,0.96] [0.88,0.89] [0.10,0.10]
[0.92,0.93]|[0.96,0.96] [0.98,0.98] [0.87,0.87] [0.07,0.08]
[0.94,0.95]|[0.98,0.98] [0.96,0.96] [1.00,1.00] [0.80,0.80]
[0.95,0.96]|[0.96,0.96] [0.89,0.89] [0.89,0.89] [0.92,0.93]
[0.96,0.98]|[0.94,0.95] [0.93,0.93] [0.83,0.83] [0.82,0.83]
[0.95,0.98]|[0.74,0.75] [0.80,0.80] [0.78,0.78] [0.77,0.77]
[0.96,0.98]|[0.56,0.56] [0.56,0.56] [0.58,0.58] [0.55,0.55]
[0.96,0.98]|[0.90,0.91] [0.87,0.87] [0.85,0.85] [0.75,0.75]
[0.97,0.98]|[0.94,0.95] [0.92,0.92] [0.89,0.89] [0.77,0.77]
[0.98,1.00]|[0.96,0.97] [0.92,0.93] [0.87,0.88] [0.75,0.76]
[1.00,1.00]|[0.96,0.97] [0.94,0.95] [0.82,0.82] [0.62,0.62]
[1.00,1.00]|[0.94,0.95] [0.89,0.90] [0.67,0.67] [0.40,0.41]
[1.00,1.00]|[0.89,0.89] [0.76,0.77] [0.50,0.51] [0.26,0.26]
[0.99,0.99]|[0.77,0.78] [0.57,0.58] [0.32,0.33] [0.18,0.18]
[0.98,0.99]|[0.56,0.57] [0.35,0.36] [0.15,0.16] [0.08,0.08]
```

```
[0.98,0.99]|[0.29,0.29] [0.16,0.17] [0.05,0.06] [0.03,0.03]
[0.96,0.97]|[0.09,0.09] [0.09,0.09] [0.03,0.03] [0.02,0.03]
[0.95,0.98]|[0.04,0.04] [0.02,0.03] [0.00,0.00] [0.01,0.01]
[0.80,0.85]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.70,0.75]|[0.01,0.02] [0.01,0.02] [0.01,0.01] [0.00,0.00]
[0.60,0.68]|[0.02,0.03] [0.02,0.02] [0.01,0.01] [0.00,0.00]
[0.57,0.60]|[0.03,0.03] [0.02,0.03] [0.01,0.01] [0.00,0.00]
[0.52,0.56]|[0.05,0.05] [0.04,0.04] [0.02,0.03] [0.02,0.02]
[0.46,0.40]|[0.05,0.06] [0.05,0.06] [0.05,0.05] [0.06,0.07]
[0.30,0.34]|[0.20,0.21] [0.20,0.21] [0.20,0.21] [0.21,0.22]
[0.18,0.22]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.15,0.16]|[0.97,0.98] [0.97,0.98] [0.97,0.98] [0.97,0.98]
[0.10,0.12]|[0.99,1.00] [0.99,1.00] [0.99,1.00] [0.99,1.00]
[0.15,0.18]|[0.99,0.99] [0.99,0.99] [0.99,0.99] [0.99,0.99]
[0.13,0.14]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.01,0.01]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]

 Columns 25 through 28
            [0.88,0.90] [0.89,0.91] [0.90,0.92] [0.91,0.93]
-----------------------------------------------------------
[0.01,0.01]|[0.95,0.95] [0.95,0.96] [0.97,0.97] [0.98,0.99]
[0.70,0.72]|[0.96,0.96] [0.96,0.97] [0.98,0.98] [0.98,0.99]
[0.73,0.75]|[0.99,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.75,0.76]|[0.00,0.00] [0.03,0.04] [0.05,0.06] [0.07,0.07]
[0.78,0.80]|[0.00,0.00] [0.02,0.02] [0.04,0.04] [0.05,0.05]
[0.88,0.89]|[0.00,0.00] [0.00,0.00] [0.02,0.03] [0.03,0.04]
[0.89,0.90]|[0.01,0.01] [0.00,0.00] [0.00,0.00] [0.02,0.02]
[0.90,0.92]|[0.02,0.03] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.92,0.93]|[0.02,0.02] [0.00,0.00] [0.03,0.04] [0.00,0.00]
[0.94,0.95]|[0.03,0.04] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.95,0.96]|[0.05,0.05] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.96,0.98]|[0.79,0.80] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.95,0.98]|[0.66,0.67] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.96,0.98]|[0.50,0.51] [0.38,0.38] [0.38,0.38] [0.30,0.51]
[0.96,0.98]|[0.61,0.62] [0.34,0.34] [0.16,0.16] [0.14,0.15]
[0.97,0.98]|[0.60,0.60] [0.23,0.24] [0.12,0.13] [0.04,0.04]
[0.98,1.00]|[0.33,0.34] [0.17,0.18] [0.06,0.07] [0.02,0.03]
[1.00,1.00]|[0.26,0.27] [0.12,0.13] [0.03,0.04] [0.01,0.02]
[1.00,1.00]|[0.20,0.20] [0.09,0.09] [0.02,0.03] [0.01,0.01]
[1.00,1.00]|[0.14,0.15] [0.08,0.09] [0.02,0.03] [0.01,0.02]
[0.99,0.99]|[0.10,0.11] [0.09,0.09] [0.03,0.03] [0.01,0.02]
[0.98,0.99]|[0.09,0.09] [0.08,0.09] [0.03,0.04] [0.02,0.03]
[0.98,0.99]|[0.08,0.08] [0.09,0.09] [0.04,0.04] [0.02,0.03]
[0.96,0.97]|[0.06,0.06] [0.07,0.07] [0.03,0.04] [0.01,0.02]
[0.95,0.98]|[0.02,0.03] [0.04,0.05] [0.02,0.03] [0.01,0.02]
[0.80,0.85]|[0.01,0.00] [0.03,0.03] [0.01,0.02] [0.02,0.03]
[0.70,0.75]|[0.00,0.00] [0.01,0.02] [0.03,0.03] [0.04,0.04]
[0.60,0.68]|[0.00,0.00] [0.01,0.02] [0.03,0.03] [0.04,0.05]
[0.57,0.60]|[0.00,0.00] [0.01,0.02] [0.03,0.03] [0.04,0.05]
[0.52,0.56]|[0.03,0.03] [0.04,0.04] [0.04,0.05] [0.06,0.07]
[0.46,0.40]|[0.09,0.09] [0.10,0.11] [0.09,0.09] [0.09,0.09]
[0.30,0.34]|[0.24,0.24] [0.25,0.26] [0.23,0.24] [0.23,0.23]
[0.18,0.22]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
```

```
[0.15,0.16]|[0.98,0.98] [0.98,0.98] [0.97,0.98] [0.97,0.98]
[0.10,0.12]|[0.99,1.00] [0.99,1.00] [0.99,1.00] [0.99,1.00]
[0.15,0.18]|[0.99,0.99] [0.99,0.99] [0.99,0.99] [0.99,0.99]
[0.13,0.14]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.01,0.01]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]

  Columns 29 through 32
            [0.94,0.95] [0.94,0.96] [0.96,0.98] [0.98,0.99]
-----------------------------------------------------------
[0.01,0.01]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.70,0.72]|[0.99,1.00] [1.00,1.00] [0.99,0.99] [0.98,0.99]
[0.73,0.75]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [0.98,0.99]
[0.75,0.76]|[0.05,0.05] [0.02,0.02] [0.00,0.00] [0.00,0.00]
[0.78,0.80]|[0.03,0.04] [0.01,0.01] [0.01,0.02] [0.00,0.00]
[0.88,0.89]|[0.02,0.03] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.89,0.90]|[0.02,0.03] [0.01,0.02] [0.00,0.00] [0.00,0.00]
[0.90,0.92]|[0.01,0.02] [0.02,0.03] [0.01,0.01] [0.01,0.01]
[0.92,0.93]|[0.00,0.00] [0.01,0.02] [0.00,0.00] [0.02,0.02]
[0.94,0.95]|[0.01,0.02] [0.00,0.00] [0.03,0.03] [0.00,0.00]
[0.95,0.96]|[0.00,0.00] [0.00,0.00] [0.05,0.05] [0.01,0.01]
[0.96,0.98]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.07,0.07]
[0.95,0.98]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.96,0.98]|[0.25,0.25] [0.35,0.35] [0.49,0.49] [0.00,0.00]
[0.96,0.98]|[0.18,0.18] [0.19,0.19] [0.34,0.35] [0.00,0.00]
[0.97,0.98]|[0.05,0.05] [0.14,0.14] [0.29,0.29] [0.00,0.00]
[0.98,1.00]|[0.03,0.04] [0.14,0.15] [0.29,0.29] [0.00,0.00]
[1.00,1.00]|[0.02,0.03] [0.14,0.15] [0.29,0.30] [0.00,0.00]
[1.00,1.00]|[0.03,0.03] [0.15,0.16] [0.30,0.30] [0.00,0.00]
[1.00,1.00]|[0.03,0.04] [0.16,0.16] [0.30,0.31] [0.00,0.00]
[0.99,0.99]|[0.03,0.04] [0.15,0.16] [0.31,0.31] [0.00,0.00]
[0.98,0.99]|[0.03,0.04] [0.15,0.16] [0.30,0.31] [0.00,0.00]
[0.98,0.99]|[0.03,0.03] [0.14,0.15] [0.30,0.30] [0.00,0.00]
[0.96,0.97]|[0.02,0.03] [0.14,0.14] [0.29,0.30] [0.00,0.00]
[0.95,0.98]|[0.02,0.03] [0.14,0.14] [0.29,0.29] [0.00,0.00]
[0.80,0.85]|[0.03,0.03] [0.14,0.15] [0.29,0.29] [0.00,0.00]
[0.70,0.75]|[0.04,0.04] [0.16,0.16] [0.30,0.30] [0.00,0.00]
[0.60,0.68]|[0.05,0.05] [0.17,0.17] [0.29,0.30] [0.00,0.00]
[0.57,0.60]|[0.06,0.06] [0.18,0.18] [0.29,0.30] [0.00,0.00]
[0.52,0.56]|[0.07,0.07] [0.18,0.19] [0.30,0.30] [0.00,0.00]
[0.46,0.40]|[0.09,0.09] [0.20,0.20] [0.31,0.32] [0.00,0.00]
[0.30,0.34]|[0.22,0.22] [0.31,0.32] [0.39,0.40] [0.00,0.00]
[0.18,0.22]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.15,0.16]|[0.98,0.98] [0.98,0.98] [0.98,0.99] [0.99,1.00]
[0.10,0.12]|[0.99,1.00] [0.99,1.00] [1.00,1.00] [1.00,1.00]
[0.15,0.18]|[0.99,0.99] [0.99,0.99] [0.99,1.00] [0.99,1.00]
[0.13,0.14]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [0.99,1.00]
[0.01,0.01]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.02,0.03]

 Columns 33 through 36
            [0.99,1.00] [0.91,0.92] [0.87,0.89] [0.85,0.86]
-----------------------------------------------------------
[0.01,0.01]|[0.98,0.98] [0.99,0.99] [1.00,1.00] [1.00,1.00]
[0.70,0.72]|[0.99,1.00] [0.99,1.00] [0.92,0.92] [0.97,0.97]
```

```
[0.73,0.75]|[0.94,0.95] [0.99,1.00] [1.00,1.00] [0.94,0.94]
[0.75,0.76]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.04,0.04]
[0.78,0.80]|[0.00,0.00] [0.00,0.00] [0.03,0.03] [0.00,0.00]
[0.88,0.89]|[0.00,0.00] [0.01,0.01] [0.00,0.00] [0.01,0.01]
[0.89,0.90]|[0.00,0.00] [0.01,0.02] [0.00,0.00] [0.00,0.00]
[0.90,0.92]|[0.01,0.01] [0.00,0.00] [0.03,0.03] [0.00,0.00]
[0.92,0.93]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.94,0.95]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.95,0.96]|[0.00,0.00] [0.00,0.01] [0.00,0.00] [0.00,0.00]
[0.96,0.98]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.95,0.98]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.96,0.98]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.96,0.98]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.97,0.98]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.98,1.00]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[1.00,1.00]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[1.00,1.00]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[1.00,1.00]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.99,0.99]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.98,0.99]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.98,0.99]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.96,0.97]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.95,0.98]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.80,0.85]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.70,0.75]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.60,0.68]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.57,0.60]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.52,0.56]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.46,0.40]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.30,0.34]|[0.00,0.00] [0.00,0.00] [0.00,0.00] [0.00,0.00]
[0.18,0.22]|[0.00,0.00] [0.00,0.00] [0.01,0.02] [0.04,0.05]
[0.15,0.16]|[0.97,0.98] [1.00,1.00] [1.00,1.00] [0.94,0.94]
[0.10,0.12]|[1.00,1.00] [0.98,0.99] [0.95,0.96] [1.00,1.00]
[0.15,0.18]|[1.00,1.00] [0.96,0.96] [1.00,1.00] [1.00,1.00]
[0.13,0.14]|[0.96,0.97] [0.98,0.98] [1.00,1.00] [0.97,0.97]
[0.01,0.01]|[0.02,0.03] [0.03,0.04] [0.00,0.00] [0.00,0.00]

 Columns 37 through 40
            [0.80,0.81] [0.70,0.71] [0.40,0.42] [0.38,0.40]
-----------------------------------------------------------
[0.01,0.01]|[0.97,0.98] [0.95,0.96] [1.00,1.00] [1.00,1.00]
[0.70,0.72]|[1.00,1.00] [1.00,1.00] [0.98,0.98] [0.97,0.97]
[0.73,0.75]|[1.00,1.00] [0.94,0.95] [0.92,0.93] [1.00,1.00]
[0.75,0.76]|[0.97,0.98] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.78,0.80]|[0.99,1.00] [0.93,0.94] [1.00,1.00] [0.98,0.98]
[0.88,0.89]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.89,0.90]|[0.98,0.99] [1.00,1.00] [0.96,0.97] [0.97,0.98]
[0.90,0.92]|[1.00,1.00] [0.97,0.98] [1.00,1.00] [0.99,1.00]
[0.92,0.93]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.94,0.95]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.95,0.96]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.96,0.98]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.95,0.98]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
```

```
[0.96,0.98]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.96,0.98]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.97,0.98]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.98,1.00]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[1.00,1.00]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[1.00,1.00]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[1.00,1.00]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.99,0.99]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.98,0.99]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.98,0.99]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.96,0.97]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.95,0.98]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.80,0.85]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.70,0.75]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.60,0.68]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.57,0.60]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.52,0.56]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.46,0.40]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.30,0.34]|[1.00,1.00] [1.00,1.00] [1.00,1.00] [1.00,1.00]
[0.18,0.22]|[0.96,0.96] [0.99,1.00] [1.00,1.00] [0.99,0.99]
[0.15,0.16]|[1.00,1.00] [0.96,0.97] [0.95,0.95] [1.00,1.00]
[0.10,0.12]|[0.97,0.97] [1.00,1.00] [1.00,1.00] [0.98,0.99]
[0.15,0.18]|[0.97,0.97] [0.96,0.96] [1.00,1.00] [0.95,0.96]
[0.13,0.14]|[1.00,1.00] [0.95,0.95] [1.00,1.00] [1.00,1.00]
[0.01,0.01]|[0.03,0.04] [0.02,0.03] [0.00,0.00] [0.00,0.00]
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9. Conclusion

In this article, very new kind of interval-valued fuzzy matrix has been introduced. In this new approach, the rows and columns are taken as uncertain, whereas in IVFM they are certain. These types of fuzzy matrices can be used to handel images, fuzzy graphs, etc.

Besides, null-IVFMFRC, equality of IVFMFRCs and identity IVFMFRC are defined. Also, some binary operators are defined. Two types of complements and density of IVFMFRC are defined and studied several properties. But, product of two IVFMFRCs is not defined. At present product of IVFMFRCs is being researched with their properties. The power convergence, nilpotency, etc. can be investigated after suitable definition of multiplication. Other several operations can also be defined for IVFMFRC.

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